CollegeBoard

The Digital SAT® Suite and Classroom Practice: Math

Evidence-Based Approaches to Helping All Students Become College and Career Ready

The Digital SAT® Suite and Classroom Practice: Math

Edited and with an introduction by Jim Patterson

About College Board

College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, College Board was created to expand access to higher education. Today, the membership association is made up of over 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success—including the SAT®, the Advanced Placement® Program, and BigFuture®. The organization also serves the education community through research and advocacy on behalf of students, educators, and schools.

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College Board
July 2023

The Digital SAT® Suite and Classroom Practice: Math aims to inform secondary math teachers (and teachers of other subject areas in which math is commonly used) of evidence-based instructional practices supporting college and career readiness for all students. The four chapters in this guide, covering topics in algebra, advanced math, problem-solving and data analysis, and geometry and trigonometry, were authored by experts in their fields who drew on both their own knowledge and experience as well as a wealth of high-quality research, citations to which can be found in each chapter's references list. In addition, College Board math content experts have created sidebars for each chapter highlighting additional evidence from College Board research as well as drawing connections between the material presented in the chapters and how the concepts discussed are commonly assessed on the digital SAT Suite of Assessments, which comprises the SAT college admission test and the PSAT/NMSQT®, PSAT™ 10, and PSAT™ 8/9 exams. (For more information about the digital SAT Suite, please visit **sat.org/digital**.)

The structure of each chapter in this guide is similar. Each author begins by discussing evidence from research and practice supporting the importance of their topic (e.g., algebra) to college and career readiness for all students. Following that, the author provides a rich analysis of how the concepts they have introduced can be implemented in the classroom. This latter discussion is generously supported with interesting and accessible samples drawn from a wide variety of professional sources. The College Board–authored sidebars previously mentioned round out each chapter's discussion with connections to the digital SAT Suite.

Companion Resource

The Digital SAT Suite and Classroom Practice: English Language Arts/Literacy (**satsuite.org/classroom-practice-english**) is also available as a free digital resource from College Board. Originally published in 2020, the 2023 version has been updated to draw connections between the chapters' discussions and the digital SAT Suite.

It is important to note that none of the four chapters is intended to provide anything like a comprehensive treatment of how to teach a given topic. Rather, the authors, individually and collectively, have focused on a small number of core concepts or principles that can and should be enacted or reinforced in secondary-level math classrooms. While the primary audience for this guide is secondary-level math teachers, the information contained herein is also likely to be of interest to teachers in other subject areas in which math is commonly used, such as in the physical and social sciences. Curriculum coordinators, school and district leaders, and policymakers are also likely to benefit from the authors' evidence-grounded discussions.

The "College and Career Readiness" sidebar in each of the four chapters makes reference to the *College Board National Curriculum Survey Report 2019*. This report presents the results of the most recent of College Board's series of surveys of secondary teachers' instructional practices and postsecondary instructors' views of prerequisites for success in first-year, entry-level, credit-bearing courses. In brief, every few years College Board asks secondary teachers and postsecondary instructors to identify what skills and knowledge the former are stressing in their classroom teaching and what skills and knowledge the latter expect incoming students already to possess to be ready to succeed in their classrooms. The basic mode of response in both cases is a four-point rating scale (with 4 being high importance/emphasis) associated with lists of skill/knowledge survey items in both ELA/literacy and math. Ratings from individual educators are averaged, and these mean importance/emphasis ratings yield evidence of what postsecondary instructors consider essential for incoming first-year students to already know and be able to do and of what secondary teachers are stressing in their lessons. Overall, evidence from this survey strongly supports College Board's claims that the digital SAT Suite tests assess key postsecondary prerequisites in ELA/literacy and in math; that the key digital SAT Suite design elements discussed in this collection are highly valued by educators; and that the digital SAT Suite is well aligned with important secondary instructional emphases. The full report may be found at **https://satsuite.collegeboard.org/media/pdf/national-curriculum-survey-report.pdf**.

Overview of This Collection

Chapter 1, by Jon Star, covers the topic of **algebra**, with particular attention to the foundational algebra of a first-year, secondary-level course. Star observes that the teaching of algebra has received renewed attention in recent years—a development clearly warranted given that, as the author asserts, "lack of access to or success in high-quality Algebra I instruction is a significant and, arguably, insuperable barrier to students' academic success in high school, postsecondary education, and well-paying careers." Star describes two different but complementary conceptions of algebra: one focused on functions and the other on equations and expressions. After detailing each conception separately, Star illustrates how the two views can and should be integrated productively in the teaching and learning of introductory algebra so that students are better prepared both for more advanced math and for college and careers.

Chapter 2, by Chris Rasmussen, delves into the topic of **advanced math**, which, as we have defined it here, focuses centrally on nonlinear equations and functions, an understanding of which builds on the foundations laid in students' study of linearity in Algebra I. Rasmussen makes a strong, evidence-based case for the relevance of advanced math skills and knowledge for all students, not just those intending to enter STEM fields. In a manner similar to chapter 1's discussion of algebra, Rasmussen unpacks two complementary conceptions of *function*: a *correspondence view* and a *covariation view*. In the former, "students are taught to consider a function as an operation for which one inputs a number and then another number, the output, emerges"; in the latter, "a function is viewed as two quantities that vary together." The author then identifies four themes that apply across all function types: "evaluating functions and solving equations, interpreting functions in context, making connections across representations, and analyzing families of functions." Rasmussen concludes with implementation advice, which counsels using authentic tasks and having students "actively engaged in questioning, struggling, reasoning, communicating, making connections, problem-solving, and explaining."

Chapter 3, by Anna Bargagliotti, addresses the topic of **problem-solving and data analysis**, particularly with respect to statistics and probability. After discussing how important it is for all students in this data-driven world to become statistically literate, Bargagliotti describes the Statistical Problem-Solving Process outlined in the *Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II)* report, for which she was the lead author and which was commissioned and published by the American Statistical Association and the National Council of Teachers of Mathematics. This process—formulate statistical investigative questions, collect/consider the data, analyze the data, and interpret the results—foregrounds active student inquiry into problem-solving and data analysis. Bargagliotti then identifies three important statistical practices: asking questions, thinking multivariately and using multiple variables, and connecting probability to statistics. The author includes numerous examples and outlines of activities throughout the chapter.

Chapter 4, by Erin E. Krupa, explores the topic of **geometry and trigonometry**. Krupa contends that "the value of geometry in K–12 education extends beyond the typical merits of understanding a subject to helping lay the foundations for achievement in other branches of math" and describes a Geometric Reasoning Cycle whose key steps—investigation, conjecture, justification, and proof—place students at the center of their own mathematical learning. The author then turns to the concepts of geometric modeling, which she characterizes as "important for applying knowledge of geometry to real-world situations," and of measurement, a critical aspect of modeling. Krupa concludes the chapter with two extended examples that integrate and implement the concepts previously discussed.

Unifying Themes

As this overview suggests, several common themes emerge across the four chapters in this collection. Among the most significant are the following:

- "Math" is not strictly (or perhaps even primarily) an academic exercise but also a means of analyzing and better understanding the world around us. Highlighting this fact is likely to prove motivating to students, who might otherwise find the subject dry, meaningless, or unmoored from real-life concerns.
- Students must be active learners of math, and to fully engage them in their studies and to expose them to the power, utility, and beauty of math, teachers must present them with authentic tasks and work with them to problem-solve.
- Central to the areas of math discussed in this guide is the concept of reasoning. While learning algorithms and set approaches has value for students, the core activity of math is engaging in such activities as questioning, analyzing, conjecturing, and justifying, whatever the subfield or course of study. An emphasis on reasoning puts students' own agency at the forefront of math learning.
- An understanding of math is cumulative, and connections can and should be drawn and reinforced between and among math's subfields. This is illustrated most explicitly in this guide's articulation of the connections between algebra and advanced math, but it carries over into all the authors' discussions of their topics' links to other areas of math.

Acknowledgements

In addition to the authors of the chapters in this guide, I would like to take this opportunity to acknowledge several individuals for their critical contributions to this work. Bill McCallum, cofounder and CEO of Illustrative Math and University Distinguished Professor Emeritus of Mathematics at the University of Arizona, was instrumental in gathering the group of authors and also provided extensive support for and review of the chapters in this collection. Dona Carling, Thom Gleiber, and Sonia Wilson, math content experts from College Board, offered essential help in planning for, reviewing, and commenting on early drafts of the chapters as well as authoring supplementary text. Sherri Miller, vice president of College Board's Assessment Design and Development department, also reviewed and commented on chapter drafts and provided overall support for the project. And, last but not least, Nancy Burkholder, Beth Oxler, and Georgina Keenan gave, as always, their best efforts in the area of editorial and design services.

CHAPTER 1

Algebra

By Jon Star

Jon Star is a professor of education at the Harvard Graduate School of Education as well as a former (and current) middle school and high school math teacher. His research focuses on the teaching and learning of algebra, with a particular emphasis on instructional and curricular interventions that promote both procedural flexibility and mathematical understanding.

1. Introduction

Algebra cuts a wide and deep swath through the secondary math curriculum. In fact, much of what most students learn in their math classrooms across the secondary grades could be classified as algebra. Students begin their study of secondary-level math in an algebra course with a focus on writing, manipulating, and using expressions and linear functions as they investigate ways to use them as tools for describing and modeling situations that are readily accessible to them. For clarity and focus, then, this chapter employs a working definition of *algebra* that focuses on topics typical of a first-year algebra course, which attends in depth to the concepts and skills associated with linear expressions, equations, and functions. Nonlinear equations and functions—a key focus of chapter 2's discussion of advanced math—are more conceptually complex than linear ones, and a grasp of the former naturally builds on a solid understanding of the latter.

There has been substantial and sustained interest in promoting students' success in algebra for a generation or more. In large-scale studies (e.g., Adelman 2006; Gamoran and Hannigan 2000; Lee and Mao 2021; Trusty and Niles 2004), success in algebra has been linked to increased secondary and postsecondary course taking, improved high school and college graduation rates, and more productive job and career outcomes. For instance, in examining nationally representative longitudinal data to study the long-term educational and career trajectories of students who were enrolled in tenth grade in 1980 (i.e., presumptive members of the high school graduating class of 1982), Rose and Betts (2001, xix) found that “math curriculum is strongly related to student outcomes more than 10 years later,” including college graduation rates and earnings. Notably, Rose and Betts found that “the biggest difference [among student outcomes] is between courses at or above the algebra/geometry level and courses below the algebra/geometry level” (xix–xx), by which they meant vocational math and prealgebra.

Recognition of the importance of strong skills in algebra has led to efforts to incorporate algebraic thinking into the elementary school curriculum (e.g., Kieran et al. 2016). Furthermore, while taking first-year algebra (Algebra I) was historically a part of the ninth-grade curriculum, many students now take this course in eighth grade or earlier (Stein et al. 2011). In almost all districts, a passing grade in Algebra I is a requirement for high school graduation.

Facility with algebra opens many doors for students; lack of such facility carries the significant risk of keeping those doors shut, whether we consider educational or vocational aspirations. Mastering concepts taught in algebra courses is viewed as a key prerequisite on the path to higher-level math courses, particularly calculus (e.g., Kaput 1995; National Mathematics Advisory Panel 2008; Rakes et al. 2010; Stein et al. 2011). Biag and Williams (2014) extend this value proposition further by noting that students failing Algebra I (and potentially having to retake it) are in danger of being cut off from advanced high school science coursework, given those courses’ algebra prerequisite. Lack of algebra skills and knowledge can also inhibit or exclude students from pursuing a range of well-paying blue- and white-collar jobs, including, among others, careers as carpenters, electricians, millwrights, and sheet metal workers as well as actuaries, architects, dietitians and nutritionists, and market research analysts (Weedmark 2018).

In short, lack of access to or success in high-quality Algebra I instruction is a significant and, arguably, insuperable barrier to students’ academic success in high school, postsecondary education, and well-paying careers. Algebra, and specifically the Algebra I course, is, therefore, an enormously important milestone in students’ math learning.

But what is algebra? There are many answers to this question, but all emphasize the role of generalization, abstraction, and symbolization within this math domain. Broadly speaking, there are two common and complementary conceptions of algebra (e.g., Lloyd, Herbel-Eisenmann, and Star 2011). First, algebra is an exploration of functional relationships. Within this view, algebra can be seen as a useful tool for modeling real-life phenomena. We use algebra and representations of algebraic functions such as graphs, symbols, and tables to analyze, describe, and make predictions about the contextual situations where quantities may vary. Second, algebra is generalized arithmetic. Instead of working with numerical expressions and equations, as is the focus in arithmetic, algebra is centrally concerned with learning to work with symbols that stand for numbers and with transformations of symbolic expressions and equations. In this sense, algebra is useful for writing and manipulating symbolic expressions and equations in order to enable both exploration and generalization of relationships between quantities that vary.

Since at least the late 1950s, the Algebra I curriculum has focused on both of these conceptions of algebra—one that foregrounds functions and the other that foregrounds equations and expressions (e.g., Herscovics 1989; Markovits, Eylon, and Bruckheimer 1986). Although, mathematically, both versions of algebra are complementary and necessary and although these same two views are well integrated in advanced math, curricula in the Algebra I course have been challenged to present students with an integrated conception of algebra (Chazan 2000). Chapters in most Algebra I textbooks can be easily categorized into those addressing topics related to equations and expressions (such as linear equations, linear inequalities, and systems of equations and inequalities) and those related to functions (particularly linear functions but progressing to quadratic and exponential functions in more advanced algebra courses). As a result, we begin this chapter by exploring separately the learning priorities for an algebra of equations and expressions and an algebra of functional relationships before concluding with recommended teaching and learning practices that can serve to promote integration of these two views of algebra.

2. Learning Priorities in Algebra When Equations and Expressions Are Foregrounded

2.1. Equivalence. A first learning priority in algebra when equations and expressions are foregrounded is to use equivalence as a bedrock principle on which all work with equations and inequalities is built (Kieran and Sfard 1999; Knuth et al. 2005; Zwetzschler and Prediger 2013). Students' study of algebra typically begins (often in a prealgebra course) with the introduction of *expressions* (e.g., $2x + 6$), which are defined as combinations of constant and/or variable terms. (A *variable* is a symbol that can stand for a varying quantity.) If we know the value of the variable(s) in the expression, we can evaluate the expression to determine its value. For example, in the expression $2x + 6$, when the value of the variable x is 7, the value of the expression is 20. Students are typically introduced to a few algebraic manipulations that can be performed on expressions—particularly the use of the distributive property and combining like terms—before they move on to linear equation solving.

Here, we suggest that algebraic expressions can serve as a more central and foundational concept in students' work in algebra. A common concern about students' learning of algebra—and equation solving in particular—is that students lack understanding of what they are doing when they are moving symbols around (Arcavi 2005). Math educators have long worried that symbol manipulation in algebra is, for many, a meaningless game of pushing *x*'s around in seemingly arbitrary ways. Evidence exists that learning can be enhanced when there are robust connections between algebraic concepts and procedures (e.g., Rittle-Johnson, Schneider, and Star 2015). The idea of equivalence can serve as the conceptual grounding for the symbol manipulation work in algebra. Equivalence is foundational to all future work in algebra equation solving as well as for all work with algebraic functions.

Using equivalence as a foundational principle for equation solving begins with the idea that there exist symbolic transformations on an expression that generate an equivalent expression. Common algebraic transformations such as the distributive property and combining like terms (the latter of which is itself an instance of the distributive property) have the effect of producing a different expression but one that has the same value as the original expression. Consider the expression $2y+4y+6$. Combining the like terms $2y$ and $4y$ yields the expression $6y+6$, which is equivalent to $2y+4y+6$. The same can be said for the distributive property, as (for example) the two expressions $6z+6$ and $6(z+1)$ are equivalent.

Equations can be viewed as statements of equality between two expressions. For example, the equation $2x+3=11$ suggests that for some value(s) of the variable x, the value of the expression $2x+3$ is equal to 11. (In this case, these two expressions have the same value, 11, when the value of x is 4.) Solving an equation is the process of finding the value(s) of the variable for which two expressions have the same value. And, importantly, when we find such a value, checking our solution means confirming that this value of the variable does indeed make the two expressions have the same value. As another example, the equation $3p-1=5p+7$ invites us to find value(s) of the variable p when the expression $3p-1$ has the same value as the expression $5p+7$. Solving this equation means trying to find this value or values; in this case, when p is -4, each expression has the value -13.

The framing of equations as statements of equality between two expressions becomes a constant presence guiding students' work into increasingly complicated equation solving. For example, early in the Algebra I course, students learn that when two expressions are equivalent, a few powerful transformations can preserve the equality of these two expressions. In particular, when students add the same value to both expressions in an equation or multiply each expression by the same amount, this action does not change their equivalence. Consider the equation $5x-7=2x-1$. This equation indicates that the expressions $5x-7$ and $2x-1$ have the same value at certain value(s) of x. (By solving this equation, we can determine that such a value of x is 2 and that when x is 2, each of these expressions has a value of 3.) When we add the constant 10 to both expressions, we end up with two new expressions, $5x+3$ and $2x+9$, but adding 10 to both sides does not change the value of x that makes these two expressions equivalent: it continues to be 2, and each expression has a value of 13 when x is 2.

Leveraging the important idea of equations as statements of equality between two expressions can also help students understand the conditions under which equation solving yields zero, one, or many solutions. Many of the equations encountered in an Algebra I course have a single solution; in such equations, the two given expressions have the same value for exactly one value of the variable. For example, in the equation discussed above, $5x-7=2x-1$, the expressions $5x-7$ and $2x-1$ *sometimes* have the same value—namely, only when x is 2. In other equations, however, there may not be any value of the variable that results in the two given expressions having the same value. Consider the equation $2x+3=2x+5$. In this equation, the expressions $2x+3$ and $2x+5$ *never* have the same value, and thus this equation has no solution. Finally, in some equations, the two given expressions *always* have the same value. For example, consider the equation $5x+8=5x+5+3$. In this equation, the expressions $5x+8$ and $5x+5+3$ are *always* equivalent, and thus this equation has infinitely many solutions.

Generally speaking, equation solving can be conceived of as the process of iteratively applying the two types of transformations mentioned above: one type that operates on a single expression and produces an always-equivalent expression (such as combining like terms and using the distributive property) and another type that preserves the equality of the two expressions in an equation (such as by adding the same value to both sides of an equation or by multiplying both sides by the same amount). Both types of transformations—or procedural actions—are helpful in equation solving because of the concept of equivalence. Thus, it benefits students to make explicit this underlying concept—in other words, to use equivalence as the conceptual glue that grounds the symbolic manipulation of equation solving.

Additional transformations of both types may be less commonly used in an Algebra I course but play a significant role in advanced math. For the first type of transformation (applied to an expression to produce another always-equivalent expression, such as by combining like terms), a special form of zero can be added to an expression. When completing the square for the quadratic expression x^2+6x, for example, it is useful to add zero in the form of $(9-9)$ to this expression by writing the equivalent expression $x^2+6x+9-9$, which can then be written as $(x+3)^2-9$. For the second type of transformation (applied to two expressions, preserving their equality), additional transformations may include squaring or taking the square root of both expressions (although more advanced work in algebra uncovers limits to how, when, and why these transformations are successful in preserving the equality of two expressions). Even as advanced algebra courses move into more intricate equations and the more complex process of solving quadratic and exponential equations, students can continue to spiral back to the bedrock principle of equations as statements of equality between two expressions by checking that solutions obtained when solving equations do indeed result in the equality of the two given expressions.

2.2. Flexibility in the Use of Strategies. A second priority for student learning of algebra when equations and expressions are foregrounded is flexibility in the use of multiple strategies, which centrally involves establishing and leveraging connections between mathematical structure and equation-solving strategies. An emphasis on multiple strategies has been present in the math curriculum in the elementary grade for some time (e.g., Silver et al. 2005), but only more recently has the field advocated the benefits to algebra students of engaging with multiple ways of solving equations. All equations can be solved in multiple ways, with some ways arguably better than others. Flexibility in the use of strategies should be seen as an instructional goal in algebra, where *flexibility* refers to knowing more than one way to solve a problem and selecting the most appropriate strategy for a given problem and problem-solving circumstance (Star 2005).

Powerful and efficient algorithms exist for solving many types of algebraic equations, and the curriculum is often focused on channeling students toward reliance on quick, automatic execution of such algorithms. For example, most textbooks push students to look for opportunities to use the distributive property as a first step to solving an equation such as $3(x+4)=12$ or $2(x+1)+5(x+1)=14$ even when alternative first steps, such as dividing by 3 in the first example or combining the like $(x+1)$ terms in the second, might be better approaches in some cases. Developing a repertoire that includes multiple approaches for solving equations, including but not limited to the application of standard algorithms, serves both to maintain students' connection to the underlying principles of equation solving (such as equivalence of expressions and the two types of equivalence-related transformations discussed above) and to support students' ability to persevere when faced with equations that are not easily solvable using standard approaches.

Furthermore, as illustrated by the examples above, consideration of alternative approaches for solving equations can require students to focus on the mathematical structure of the symbolic expressions they encounter and to establish links between structural features of expressions and equation-solving strategies that may be possible or productive. *Structure* of an expression refers to its underlying symbolic and mathematical features, such as the position of and relationships between quantities and operations. Using the first example above, $3(x+4)=12$, noticing that 12 is evenly divisible by 3 is a recognition of structure that may distinguish this equation—and the strategies that might be advantageous for solving it—from an equation such as $3(x+4)=11.2$. In the second example, $2(x+1)+5(x+1)=14$, noticing the repeated presence of $(x+1)$ terms is another instance of recognizing structure that can point to strategies for solving this equation that may not be possible or productive with similar problems, such as $2(x+3)+5(x+1)=14$. Recognizing structure and making links between structure and solving strategies are enormously powerful tools as students reckon with increasingly complex equations (Star et al. 2015). A focus on structure encourages students to reflect on the features of an equation that suggest a given strategy is applicable or might be especially useful.

Let us turn to an example where a focus on flexibility in the use of strategies is already present in the curriculum: the solving of systems of linear equations. Most Algebra I textbooks commonly used in the United States emphasize three different ways to solve linear systems: *elimination*, *substitution*, and *graphing*. Not only are students expected to develop proficiency in all three of these strategies, but also many curricula push students to consider the structural features of a linear system that might make a particular strategy preferable. Consider the following system of linear equations:

$$3x-4y=12 \qquad (1)$$

$$7x+4y=8 \qquad (2)$$

In this case, because the y-coefficients of equations (1) and (2) are additive inverses, the elimination method could be considered optimal. Adding the two equations together yields $10x=20$, allowing for an efficient way to solve for x. But consider this second system of linear equations:

$$y=2x+1 \qquad (3)$$

$$4x-5y=10 \qquad (4)$$

In this case, since equation (3) is already solved for the variable y, it might be most appropriate to use the substitution method. This would yield the new equation $4x-5(2x+1)=10$, which can be solved for x.

However, as these examples suggest, the identification of an optimal strategy for a given linear system may not be unambiguous or universally agreed on. A given system may have several of the structural features identified above, thus suggesting multiple and conflicting possibilities for an optimal strategy. Furthermore, students' facility with and/or preference for a particular strategy may influence their determination of which strategy is best. More generally, what makes one strategy more appropriate for a given problem will often require some discussion and the resolution of differing opinions.

Digital SAT Suite Connections

Coursework in algebra is very important in each student's math journey, and facility with algebra provides students with opportunity for further success, while lack of facility burdens them with risk of reduced opportunity. As a result, skills in algebra have significant representation on the digital SAT Suite exams, representing approximately 42 percent (PSAT 8/9), 35 percent (PSAT/NMSQT and PSAT 10), and 33 percent (SAT) of the test content. Questions in the Algebra content domain of each exam align most closely with topics covered in a typical rigorous first-year secondary algebra course, including assessing the skills and knowledge associated with working with linear expressions, equations in one and two variables, functions, systems of equations, and inequalities. Test questions cover such skills and knowledge as creating and using an equation; identifying an expression or equation that represents a situation; interpreting parts of an equation in context; making connections between equations, graphs, tables, and contexts; determining the number of solutions and the conditions that lead to different numbers of solutions; and calculating and solving. The test questions aligned to algebra skill/knowledge elements range in difficulty from relatively easy to relatively complex and challenging. The test questions require students to demonstrate skill in generalization, abstraction, and symbolization, with a strong emphasis on equivalence and using structure. Many of the test questions are constructed to allow for more than one solving strategy.

Instructionally, discussions of multiple strategies for solving problems and the relative merits of particular approaches in given cases can be productive means of developing flexibility as well as student engagement. Discussions about multiple ways of solving algebra problems can resemble the discussions that might occur in a higher-level math class on, say, the elegance of a particular proof. These types of discussions are deeply mathematical and often involve a mixture of objectively determined (structural) and subjectively determined (aesthetic) considerations. Participation in such discussions can be a mathematically authentic and important part of students' learning in algebra and need not be reserved for more advanced students.

3. Learning Priorities When Functions Are Foregrounded

3.1. Use of Multiple Representations. Within a view of algebra that foregrounds functions, a central learning priority for students is the ability to use multiple representations of functions (expressions of relationships between quantities that vary) to model and analyze contextual situations. Representations of functions allow us to explore generality in such relationships as well as the relationships between the behavior of the functions' independent and dependent variables. Commonly used representations of functions include tables, graphs, and expressions and equations.

Consider the context of a gym that charges an initial fee of \$10 to join plus a fee of \$15 each month. We can use multiple representations of functions to model and explore this context. Symbolically, after m months, the total cost, c, of the gym membership can be written as $c = 10 + 15m$. This relationship can also be depicted using a table or a graph, as shown below.

Number of months	Total cost, in dollars
1	25
2	40
3	55
4	70

Table 1. Gym membership cost (tabular form)

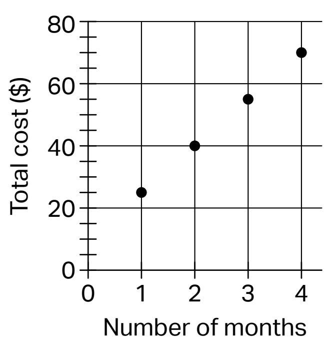

Figure 1. Gym membership cost (graphical form)

Establishing and using links between multiple representations of functions have many important benefits for students' learning of math. First, links between multiple representations can illuminate aspects of the contextual situation that is being modeled—particularly patterns and rates of change—which can facilitate analysis and prediction in the situation. Patterns and rates of change can themselves be depicted in distinct but complementary ways by different representations. In graphical representations of a function, rates of change can be seen in the “movement” of the graph of the function, particularly in terms of how steeply and in what direction the curve rises or falls. How variables change can be analyzed in tabular representations by examining ways that the values in one row or column are different from the values in another row or column. Students come to learn that certain types of change are readily apparent in the symbolic expressions we write for particular functions, while other forms of change take a bit of symbol manipulation to make evident.

Returning to the gym example above, we observe that patterns of change are depicted in very different ways across the three forms of representation: tabular, graphical, and symbolic. In the table, we can observe several different types of change relationships. For example, when examining how the “Total cost, in dollars” column relates to the “Number of months” column, we might notice that in the first row the total cost is 25 times the number of months but that in the second row the total cost is 20 times the number of months. (This observation, while valid, is not especially useful in helping us analyze this contextual situation or recognize its linearity.) Alternatively, we can see how the “Number of months” column changes (increases by 1 in each row) as well as how the “Total cost, in dollars” column changes (increases by 15 in each row). Using the graph, we might perform computations to quantify the steepness of the line that take into account the line’s vertical and horizontal change from one point to another point. We might also notice the similar right triangles created from any two pairs of points. In the symbolic representation of this situation, $c = 10 + 15m$, both the 10 and the 15 reveal important information about patterns of change in the relationship between the number of months and the total cost. Making connections across these forms of representation allows us to identify similarities and differences between these different depictions of change as well as enrich our understanding of change in this particular context and how change in general can be variously depicted in tables, graphs, and symbols.

Second, in addition to allowing learners to better understand important aspects of the specific contexts being modeled, links between multiple representations also illuminate aspects of the math embedded within the functions. By looking at the same function through graphs, tables, and symbols, students make use of differing lenses through which to examine the function, allowing them to notice how the features observed in one representation are also observable in a different representation of the same function.

Consider how links between multiple forms of representation may help students recognize what happens to a function, $f(x)$, when 1 is added to the function and when 1 is subtracted from the function. When students first encounter these types of transformations in an Algebra I course, it can be quite confusing to understand whether (and why) adding to or subtracting from the value of a function has the effect of translating the graph up, down, to the left, or to the right. These distinctions are especially difficult to understand when we consider linear functions, as the same transformation can be viewed both ways: as a vertical or a horizontal transformation. In figure 2, note that because, for this particular function, $f(x+0.5)=f(x)+1$, $g(x)$ (leftmost graph) can be viewed either as a transformation of $f(x)$ up one unit or a transformation of $f(x)$ one-half unit to the left. Similarly, because $f(x-0.5)=f(x)-1$, $h(x)$ (rightmost graph) can be viewed as either a transformation of $f(x)$ down one unit or a transformation of $f(x)$ one-half unit to the right.

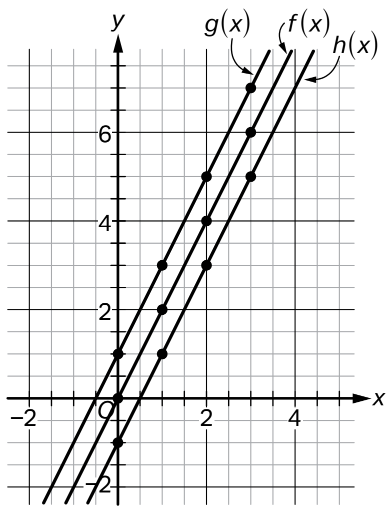

Figure 2. Linear transformations of function *f*(*x*) (graphical form)

However, by linking the graphical and tabular representations of these functions, the vertical transformations that result from $f(x)+1$ and $f(x)-1$ are easier to recognize. Table 2 indicates that to obtain the listed values for $g(x)$, we add 1 to each value of $f(x)$; on the graph, this same "movement" from each point on the graph of $f(x)$ to the corresponding point on the graph of $g(x)$ can also be observed.

x	$f(x)=2x$	$g(x)=f(x)+1$	$h(x)=f(x)-1$
0	$f(0)=0$	$g(0)=1$	$h(0)=-1$
1	$f(1)=2$	$g(1)=3$	$h(1)=1$
2	$f(2)=4$	$g(2)=5$	$h(2)=3$
3	$f(3)=6$	$g(3)=7$	$h(3)=5$

Table 2. Linear transformations of function *f*(*x*) (tabular form)

As students progress beyond algebra and into advanced math courses, the aspects of functions that they study become more complex and diverse. They may be asked, for example, to determine where a function reaches its maximum or minimum value or other specified values (e.g., roots), to examine the behavior of the function for very large or very small values, or to ascertain whether certain regularities or symmetries exist in the behavior of the function more generally. It becomes increasingly useful in these advanced courses to leverage and link several representations in order to get a complete mathematical picture of the behavior of functions.

The algebra curriculum is full of opportunities to leverage connections across forms of representation as a way to improve students' understanding of both the context in which a math problem is situated and the math embedded within the context. Take, for instance, the notion of slope, a central concept in the investigation of linear functions that both builds on work in proportional reasoning undertaken in earlier grades and foreshadows the more general interest in rates of change employed in advanced math. Slope—and, more generally, rates of change in linear relationships—is depicted in differing ways in tables, graphs, and symbols. Students benefit from drawing connections across these forms of representation. In later coursework, students' understanding of quadratic functions in advanced math is enhanced by making these same kinds of connections across forms of representation.

Third, linkages between multiple forms of representation are useful in building students' understanding of the critically important concept of function more generally. Functions are often introduced to students using an input/output metaphor, in which a function is conceptualized as a rule that describes the relationship between an input (or independent variable) and an output (or dependent variable). We can not only look at the ways that input/output relationships are variously represented in symbols, tables, and graphs, but we can also investigate patterns of change in the inputs and outputs across these representations. For example, what happens to the output when there is a one-unit change in the input, and how is this variously depicted in tables, graphs, and symbols? Alternately, what happens to the input when there is a one-unit change in the output, and how is this shown in different representations? The central idea of a function—that for a given function, each input is linked to exactly one output—is enhanced when it is illustrated and when links are established across multiple forms of representation. Graphically, the vertical line test (wherein a vertical line drawn anywhere on the function goes through at most one point) is commonly used to illustrate the definition of a function, yet a deeper understanding of the concept of function can be achieved if students can relate this graphical test to a table displaying the function wherein each input has a unique output.

CollegeBoard

College and Career Readiness

The College Board National Curriculum Survey Report 2019 offers clear indications that algebra skills and knowledge are important college and career readiness prerequisites.

The sample of 268 postsecondary math faculty rated the following algebra skill/knowledge elements assessed on the SAT Suite on a four-point scale, with 4 meaning "very important":

- Represent contexts using a linear expression or equation in one variable (3.67), a linear equation in two variables (3.67), a linear inequality in one or two variables (2.82), a system of linear equations (2.93), and a system of linear inequalities (2.39)
- Interpret variables, constants, and/or terms in a linear equation (3.73)
- Evaluate a linear expression (3.71)
- Solve a linear equation (3.77) and a system of two linear equations (2.96)
- Graph a linear equation (3.71), a linear inequality (3.06), a system of two linear equations (2.86), and a system of two linear inequalities (2.30)

Ratings for all but two of these skill/knowledge elements met or exceeded the threshold of 2.50 established as indicating a given skill/knowledge element was considered important as a prerequisite for readiness for postsecondary math.

For more information on College Board's national curriculum survey and its results, see the general introduction to this collection.

4. Establishing Connections between an Algebra of Equations and Expressions and an Algebra of Functions

To this point, a series of learning priorities has been proposed for algebra when this domain is conceived of as (a) the study of equations and expressions and (b) the study of functions. It is perhaps developmentally appropriate for these two foci to be independently addressed as students first encounter algebra. But, ultimately, as in the study of advanced math, these two perspectives are fully woven together, and so we now consider how and why the process of integrating these two perspectives can begin even in an introductory algebra course. Two ways for commencing this process of integration are described below. The first involves explicitly establishing bridges between equation solving and functions in algebra; the second involves conceptualizing the algebra course as a coherent, cyclic progression through increasingly complex types of functions.

4.1. Building Bridges between Equation Solving and Functions. Once students have developed a basic understanding of and fluency with algebra as equations and expressions as well as algebra as functions, it becomes important to identify and take advantage of opportunities to begin to build bridges between these two perspectives. Such connections can go in either direction.

One powerful way that functions and equation solving can be connected is through graphical visualization of the equation-solving process (Yerushalmy and Chazan 2002). Consider the equation $2x+3=5x-3$. Our suggestions above were to consider this equation as stating the equality of the expressions $2x+3$ and $5x-3$ and our task as finding the value of the variable x such that both expressions have the same value. (Note that, in this case, this value of x is 2 and that both expressions have a value of 7 when x is 2.) By graphing the function $f(x)=2x+3$ and $g(x)=5x-3$, we can see in figure 3 that when x is 2, both functions have the same value of 7. The values of each function can be seen by looking at the y-coordinates of each point on the lines; the y-coordinates of both functions are the same (7) when x is 2, which is why the lines intersect. This point of intersection can also be confirmed arithmetically, since $f(2)=2(2)+3=7$ and $g(2)=5(2)-3=7$.

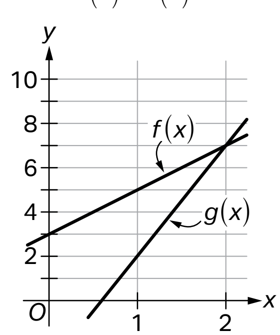

Figure 3. Graphical visualization of the equation-solving process for $2x+3=5x-3$ **using the functions** $f(x)=2x+3$ **and** $g(x)=5x-3$

Equation solving can thus be represented by graphing the two expressions in the equation and finding their point of intersection. This point of intersection can be arithmetically confirmed using the symbolic representations of the two equal expressions. This type of visual equation solving has the potential to establish connections for students between the equation-solving perspective of algebra and the functions perspective.

4.2. Emphasizing the Cyclic Nature and Coherence of the Algebra Curriculum. A second way to build bridges between the algebra of equations and expressions and the algebra of functions is to make explicit the similarities in the types of tasks that we engage in across increasingly complex functions, as this fosters coherence between the algebra course and later courses in advanced math. To counteract the possibility that students will erroneously see the foci of algebra and subsequent advanced courses as unrelated, a productive and coherent vision of the high school curriculum would involve emphasizing the similarities between the various types of functions explored in algebra and advanced math courses as well as between the types of mathematical investigations and questions that we pursue for all types of functions. For example, how we solve equations in our investigation of linear equations in the algebra course is very similar to how we solve equations for quadratics and exponential relationships in advanced math courses; we merely need additional tools for tackling the more sophisticated functions. Noticing and interpreting features of graphs as well as structural features of symbolic expressions are common to our work with all types of functions. Coherence of the math curriculum increases when we make it clear to students that we are asking them to engage over time with increasingly complicated types of functions using the same sorts of approaches they learned in earlier coursework. This perspective situates the study of algebra throughout secondary school as an investigation of “families of functions” (Confrey and Smith 1991; Schwartz and Yerushalmy 1992) that moves from working with linear functions in Algebra I to quadratic, exponential, trigonometric, logarithmic, and higher-degree polynomials in later years. The initial work around families of functions in Algebra I may be concerned with solving equations and finding x- and y-intercepts, but over time in advanced math courses we move to more sophisticated investigations that involve full integration of functions-oriented and equation-solving perspectives, including transformations and the study of inverse relationships.

5. Conclusion

The study of algebra plays an initial and critical role in students’ high school math education. Algebra includes a focus on foundational understandings of functions and functional relationships as well as the development of fluency with and understanding of the conceptual underpinnings of equations and expressions. This chapter sketched out some of the most important content and teaching practices in the Algebra I course. We recommended that the course's focus on equations and expressions include emphases on equivalence as a bedrock principle supporting all future work in equation solving as well as flexibility in the use of strategies. For aspects of algebra that foreground functions, we advocated for finding and using links between multiple forms of representation as essential learning goals. Finally, we discussed the importance of building bridges between the algebra of equations and expressions and the algebra of functions, particularly through incorporating a graphical visualization of the equation-solving process and stressing the cyclic nature and coherence of the math curriculum. Instructional emphases on these points can increase the chances that students will be successful in this essential course and beyond. ❖

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CHAPTER 2

Advanced Math

By Chris Rasmussen

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1. Introduction

Advanced math is a concept that lacks a singular definition, even if we limit our consideration to secondary instruction. This circumstance obviously arises in part because the notion of *advanced* is relative. The Algebra I course, for example, would not be considered advanced when taken by first-year high school students but might be when, as is often the case today (Stein et al. 2011), taken by eighth graders or even younger students. Another complication is that variations in course names and descriptions across the nation's schools (e.g., Conley 2007a) inevitably obscure clarity on what counts as advanced math. If we take a strictly course-based approach to definition, we might reasonably consider anything beyond Algebra I and geometry as advanced math, but any such attempt at a definition is likely to provoke dissent.

As an alternative, this chapter proposes a working definition of *advanced math* focused on a broad conceptual divide between a key focus of Algebra I and higher-level math: while Algebra I attends centrally to the concept of linear equations and functions, advanced math, as we treat it here, focuses centrally on nonlinear equations and functions. One way to quickly conceptualize this distinction is to note that linear equations and functions graph as straight lines, while nonlinear equations and functions do not. Because nonlinear equations and functions are more conceptually complex than linear ones and because an understanding of the former builds on an understanding of the latter, nonlinear properties can reasonably be categorized as "advanced." Indeed, as students progress in their study of math, they build on their earlier experiences with algebraic expressions and linear functions to investigate the ways in which nonlinear equations and functions are powerful tools for making sense of and modeling phenomena in their worlds.

Advanced math skills and knowledge, as we have defined them here, are relevant to secondary-level students in numerous ways. First, advanced math in high school serves as a bridge to still more advanced coursework in math in both high school and college, and it opens access to coursework in secondary and postsecondary science that has advanced math prerequisites. Carnevale and Fasules (2021, 1), for example, pulled together data from the U.S. Census Bureau and the Occupational Information Network, a database sponsored by the U.S. Department of Labor's Employment and Training Administration, and found that "jobs in science, technology, engineering, and math (STEM) use the highest levels of math, with 92 percent of STEM workers needing to know at least Algebra II. . . . Most STEM jobs require even higher-level math, with 67 percent requiring college-level math such as calculus." Thus, the study of advanced math is an essential pathway toward STEM-related professions.

Second, attaining advanced math skills and knowledge in high school is important for college and career readiness for students seeking entry into a wide range of blue- and white-collar occupations, both outside and, especially, within STEM fields. Based on statistical analysis of employment data as well as input from business leaders and over three hundred two- and four-year faculty, the American Diploma Project (2004) found a convergence between the knowledge and skills employers seek in new workers and those that college faculty expect of entering students. In particular, both employers and college faculty expect high school graduates to be able to apply math concepts typically taught in advanced secondary coursework in algebra. This finding for the continued work in math beyond a first course in algebra is consistent with the more recent recommendation from the report *Catalyzing Change in High School Mathematics: Initiating Critical Conversations* (National Council of Teachers of Mathematics 2018). This report recommends that high schools offer continuous four-year math pathways, including two to three years in a common pathway that includes focused attention on learning the concept of function (one of the "Essential Concepts" in high school math).

Third, acquisition of advanced math skills and knowledge is associated with positive educational and economic outcomes for students. Advanced math is typically a requirement for entry into a four-year college. For example, in an analysis of college-going California high school students, Asim, Kurlaender, and Reed (2019) found that compared to the overall population of high school seniors, a significantly larger proportion of students who applied and were admitted to either a California State University or a University of California institution took advanced math courses (for which advanced algebra is a prerequisite) in their senior year. This is consistent with prior research that found similar correlations to college entry as well as to college completion (e.g., Gottfried, Bozick, and Srinivasan 2014; Long, Conger, and Iatarola 2012). Research also has identified correlations between higher earnings and completion of more and higher levels of math (e.g., Rose and Betts 2004). Indeed, Moses and Cobb (2001) refer to algebra as the new civil right, as students who do not have access to higher-level math have less access to economic mobility.

Fourth, principles and methods of advanced math can be applied productively to analyze and understand a gamut of academic and real-world scenarios that students will encounter throughout their lives. Because many authentic applications, both within the field of math and in the real world, are nonlinear, students will need to work with quadratic, polynomial, rational, exponential, and other nonlinear functions. For example, quadratic functions are useful models for understanding and analyzing real-world situations such as forecasting business profit and loss, modeling projectile motion, and describing the movement of bouncing objects. Polynomial functions can be used to model the curves in a roller coaster, the concentration of a particular drug in the bloodstream, and other real-world situations. Rational functions are useful for analyzing real-world phenomena such as density, work, rates of change, and volume. Several examples of modeling with exponential functions are provided later in this chapter. The study of these different nonlinear function types can develop both the habits of mind and habits of interaction that students need to become powerful users of math, to better interpret and understand their worlds, and to make better predictions about phenomena of interest.

Given the importance of function in advanced math, we begin by unpacking two central meanings of *function*. This is followed by an elaboration on four themes that cut across all function types: evaluating functions and solving equations, interpreting functions in context, making connections across representations, and analyzing families of functions. We conclude the chapter with some advice for implementation.

2. Unpacking the Concept of Function

The two primary, complementary perspectives used to interpret and make sense of functions are the *correspondence view* and the *covariation view*. A *correspondence view* is one in which students are taught to consider a function as an operation for which one inputs a number and then another number, the output, emerges. This approach is quite prevalent in secondary math curricula and state standards (e.g., NGA Center for Best Practices and Council of Chief State School Officers 2010). Metaphorically, the correspondence approach is often thought of in terms of a function machine that uses a particular algebraic rule to produce the output from a given input. The relationship between the input/output pairs is then a mapping between sets, with each value in the input set (domain) mapped to exactly one value in the output set (range).

As students become more proficient in creating input/output pairs for multiple function types and as their understanding of the correspondence view deepens, they begin to conceive of the input/output process in its entirety and are able to reason about a function as an object consisting of the complete set of input/output pairings (Sfard 1991). For example, a function viewed as an object can be represented as a graph, which can then be shifted up, down, left, or right. More on the importance of such function transformations appears later in this chapter.

A second perspective of function is that of *covariation*, in which a function is viewed as two quantities that vary together. This view involves thinking of the quantities simultaneously and imagining how they change in tandem in a fluid, dynamic way (Carlson et al. 2002). Thinking about how two quantities continuously vary in relation to each other is arguably a core way of thinking about rate of change (Kaput 1994) and is one of the "big ideas" that constitute an essential understanding of function detailed by Cooney, Beckmann, and Lloyd (2010). Thus, this view of function is foundational for the math that some students will later pursue, such as calculus, in which students will continue to develop their understanding of change, including the direction of change, the amount of change, the average rate of change, and the instantaneous rate of change (Thompson and Carlson 2017).

As an example of covariational reasoning, consider a table that coordinates the input and output for some unknown function, f, such as that shown in table 1. By analyzing how the output of the function changes as the input changes, students can determine whether the function is linear or nonlinear. If nonlinear, it may even be possible to determine the particular family of functions to which the function belongs. Taking a covariation perspective on how the input and output in table 1 change together reveals that as the input increases by 1, the output decreases by a nonconstant amount. More specifically, the pattern of change between the input and output indicates that the function belongs to the family of quadratic functions. Quadratic functions are often used to model the distance an object has fallen over time.

x	$f(x)$
0	400
1	384
2	336
3	256
4	144
5	0

Table 1. Inputs and outputs for a quadratic function, *f*

Covariational reasoning can also be a powerful way to analyze contextual situations for which neither a table of values nor a function rule is provided. For example, imagine a roundish vase, such as that shown in figure 1, being filled with water at a constant rate. Without access to any equation or function rule, readers are invited to imagine what a graph of the height (H) of the water in the vase versus (i.e., as a function of) the volume (V) of water should look like and to sketch their representation of it before reading further.

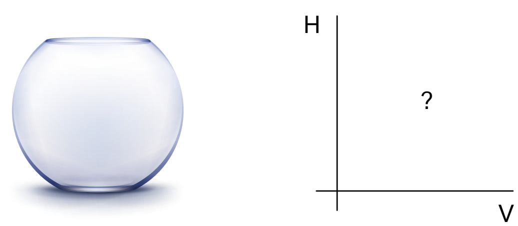

Figure 1. Water filling a roundish vase

Imagining how the height of the water changes as the volume increases likely brought to mind how fast (i.e., the rate at which) the vase fills up. Because the diameter of the vase is relatively small at the bottom and increases in size going toward the middle of the vase's height, the height of the water rises more quickly at the beginning of the filling process and then more slowly as the water approaches the middle of the vase. The opposite is true as the water fills the vase from the middle of the vase's height to the top of the vase, since the diameter of the vase is the greatest at the middle of the vase's height and decreases in size going toward the top of the vase. Thus, the height-versus-volume graph should start off with a fairly steep slope that gradually tapers off (i.e., concave down), followed by a symmetrical concave-up portion, with the change in concavity occurring at the midpoint of the graph (which corresponds to the midpoint of the vase). Another way to conceptualize the shape of the graph is by imagining using a glass to successively pour in equal amounts of water while keeping track of the increase in the water's height after each pour interval. This discrete-solution approach nicely represents the possibilities of and interplay between discrete and continuous reasoning, two ways of thinking about covarying quantities that are central to much of math and to college and career readiness (Castillo-Garsow, Johnson, and Moore 2013; Jacobson 2014).

3. Four Central Themes of Functions

We now turn to the four themes of function noted previously: (1) evaluating functions and solving equations, (2) interpreting functions in context, (3) making connections across representations, and (4) analyzing families of functions.

(1) EVALUATING FUNCTIONS AND SOLVING EQUATIONS

Curricula that emphasize a correspondence view require students to evaluate functions and move back and forth between inputs and outputs. Part of the intellectual work involved in doing this is a solid familiarity and fluency with function notation. Gaining this familiarity and fluency is not usually a onetime endeavor and will likely require students to have many encounters with multiple types of functions. Unpacking and interpreting the conventional meaning of the various symbols that appear in a function—meanings that took centuries to land on (Yoon and Thompson 2020)—is important both for communication of ideas and for individual understanding. For example, consider $f\left(x\right)=x\left(11-2x\right)\left(8.5-2x\right)$. What does the *f* refer to? What part of this equation do we associate with the output? How does this equation relate to the mapping of sets? In figure 2, adapted from Thompson and Milner (2019), the conventional mathematical meanings for each of the various components in this equation and how these meanings relate to the correspondence view of function are specified.

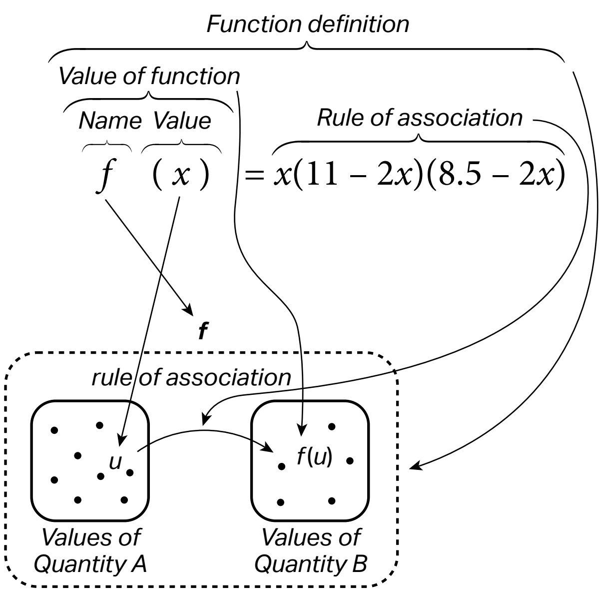

Figure 2. Meaning of terms and relationship to the correspondence view of function (adapted from Thompson and Milner 2019, 58)

When we as teachers and curriculum designers are not consistent in how we refer to and label the function (e.g., $y=x^2$, $y=f\left(x\right)=x^2$, and $f\left(x\right)=x^2$, which essentially mean the same thing), the input, or the output, it is no wonder that students often see function notation as a jumble of symbols and have difficulty translating between representations. As evidenced in the standards document put forth by the National Governors Association (NGA) Center for Best Practices and Council of Chief State School

Officers (2010), attending to precision and developing fluency in the use of notation and language are keys to success in advanced math.

Other important function competencies include the ability to associate input/output pairs, to interpret a function when it describes a context, and to make sense of inputs to functions other than x (e.g., $x+1$). These important skills go hand in hand with the development of coherent conceptual understandings of how functions assign relationships between inputs and outputs. For example, students can be given a graph of a function, f, and asked to solve $f(x)=-2$ for x. This type of task reverses the standard exercise of finding the output of a function for a given input by asking students to identify the input that would yield a given output. Such flexibility in moving between inputs and outputs helps develop coherent conceptual understanding of function relationships.

Another variation to the standard task of evaluating a function for a given numerical input is to vary the input itself. For example, given a function such as $A(s)=s^2$, for $s>0$, which is the function that provides the area of a square for possible side lengths s, students should be able to explain the relationship between $A(s)$ and $A(s+3)$. As revealed by the work of Marilyn Carlson (1998), however, even college precalculus students have difficulty evaluating something such as $f(x+a)$. Instead of viewing the expression $(x+a)$ as the input, 43 percent of students added a onto the end of the expression for f. This suggests that students were not thinking about inputs and outputs. Certainly, students need lots of opportunities to work flexibly and meaningfully with functions.

Solving and rewriting formulas or equations are other important skills that both promote and demonstrate flexibility. For example, students should be able to rewrite the formula for the area of a square in terms of its perimeter (i.e., since $P=4s \Rightarrow A(s)=P^2/16$). Rewriting formulas or equations can also help to reveal important aspects of equation structure and the connection between different forms of an equation of a function and the graph of that function, particularly for quadratic functions. For example, when one inspects a quadratic function written in standard form $f(x)=ax^2+bx+c$, the structure of this equation readily reveals that the graph of f has a y-intercept of c. It does not, however, allow one to readily figure out whether there are two, one, or no zeros (and their location) or where the vertex of the parabola is located. To ascertain these features of the graph, structuring the equation in other ways could be more useful. Making use of the factored form allows one to readily determine zeros (or roots) of the function. On the other hand, using the technique of completing the square to obtain the vertex form, $f(x)=a(x-h)^2+k$, allows one to readily determine the coordinates of the vertex of the graph of the function. Thus, moving flexibly between these different structures for a quadratic function is an important competency that allows for different insights into features of a function or its graph.

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Digital SAT Suite Connections

The advanced math topics assessed on the digital SAT Suite exams extend those covered in the Algebra content domain into nonlinear equations and functions and align most closely with topics mastered in a typical rigorous second-year secondary algebra course and sometimes beyond. Since these Advanced Math test questions build on skills first mastered with linear expressions and equations, it follows that these topics should also be well represented on college and career readiness exams such as those of the digital SAT Suite. As a result, skill/knowledge elements in Advanced Math are represented on the digital SAT Suite exams in relatively high proportions: 20 percent (PSAT 8/9) and 33 percent (PSAT/NMSQT, PSAT 10, and the SAT).

The Advanced Math content domain assesses skills and knowledge associated with working with quadratic, exponential, polynomial, rational, radical, absolute value, and conic section equations and functions. Similar to Algebra questions, test questions in the Advanced Math domain cover such skill/knowledge elements as creating and using a nonlinear equation; identifying an expression or equation that represents a situation; interpreting parts of an equation in context; making connections between equations, graphs, tables, and contexts; determining the number of solutions and the conditions that lead to different numbers of solutions; and evaluating and solving using nonlinear equations and systems that include a nonlinear equation.

(continued on next page)

These different structures for a quadratic function can also be investigated with dynamic graphing software in ways that allow learners to explore the effect of changing a parameter, make predictions, and then use algebra to confirm or revise their predictions. For example, students can use graphing software to explore how changing the value of n in the quadratic function $f(x) = x^2 + nx + 2$ affects the number and location of the x-intercepts of the graph, which correspond to the solutions to $f(x) = 0$. Students can then use the quadratic formula to precisely determine the conditions for which there are two, one, or no zeros. Observing how the graph changes as n varies also suggests that the vertex travels along a parabolic path. Students can then algebraically investigate whether this is really true. For example, since the x-coordinate of the vertex (as revealed when the equation is rewritten in vertex form) is $x = -n/2$, students can then rewrite this so that x is in terms of n, which yields $n = -2x$. Substituting $-2x$ for n into f to find the y-coordinate of the vertex yields $y = x^2 + (-2x)x + 2 = -x^2 + 2$. Thus, the vertex of the parabola travels along the path determined by $y = -x^2 + 2$. Students can then return to their graphing software to corroborate this result. This is just one example of the many possibilities for using technology with different structural forms of equations to gain insight into their different features or properties. The book *Focus in High School Mathematics: Technology to Support Reasoning and Sense Making* by Dick and Hollebrands (2011) offers many other excellent examples.

(2) INTERPRETING FUNCTIONS IN CONTEXT

Interpreting functions in context helps students develop a wide range of competencies related to modeling phenomena, which is essential for college and career readiness. Indeed, as argued by Conley (2007b, 15), "College-ready students possess more than a formulaic understanding of mathematics. They have the ability to apply conceptual understandings in order to extract a problem from a context, use mathematics to solve the problem, and then interpret the solution back into the context." Whether the contexts in which functions are interpreted are from other areas of math curriculum (e.g., properties of polygons) or from phenomena in the world around us that we seek to better understand, modeling provides opportunities for learners to investigate these situations, make predictions, understand phenomena better, solve problems, and interpret results.

Modeling with nonlinear functions requires students to recognize how patterns of change between quantities differ from those for linear functions. Recognizing these differences is usefully done by drawing on a covariation view of function. When students examine patterns of change between two quantities, an insightful and useful question for them to ask is, As one quantity increases in constant increments, by what amount does the other quantity change? (Carlson and Oehrtman

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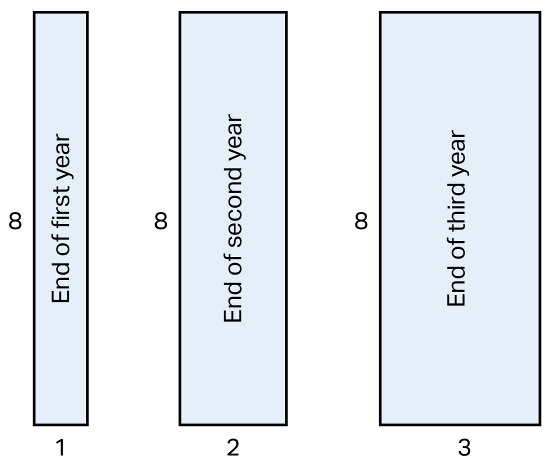
The test questions in the Advanced Math domain range in difficulty from relatively easy to relatively complex and challenging. The questions provide opportunities for students to demonstrate their skill with and knowledge of functions developed from both conceptions of function discussed in the chapter. Many of the test questions represent challenging, authentic problems in context for which students can draw on strategies developed during their coursework to solve.

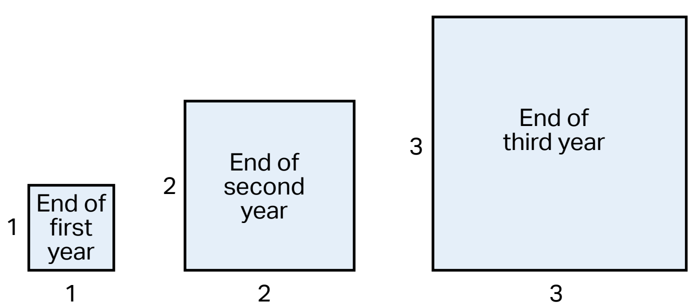
2005; Jacobson 2014). For linear functions, the dependent quantity changes by a constant amount. That is, the rate of change is constant. The corresponding big idea for linear functions is that for unit changes in the independent variable, the first difference in the dependent variable is constant. In comparison, the corresponding big idea for quadratic functions is that the rate of change of the rate of change is constant (Lobato et al. 2012). Thus, rather than the first difference being constant, the second difference is constant. The big idea for exponential functions, on the other hand, is that they model situations in which a quantity is multiplied by a constant factor for each unit period of change.

The "growing rectangles" task from the book *Tasks and Competencies in the Teaching and Learning of Algebra* by Friedlander and Arcavi (2017, 117–18) illustrates nicely how a covariation approach can be leveraged to distinguish between linear, quadratic, and exponential functions. In this task, students are given the following descriptions and illustrations for three different imaginary processes for growing rectangles over a series of years:

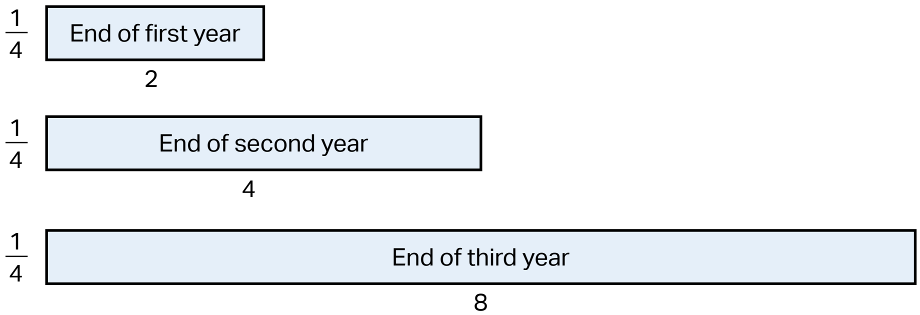
- The figure starts as a segment of 8 length units and gradually grows into a rectangle. By the end of each year, one of its sides becomes longer by an additional unit but its other side stays fixed at 8 units of length.



- The figure starts out as a point and gradually grows into a square. By the end of each year, its sides become longer by an additional unit.



- The figure starts as a rectangle of dimensions $\frac{1}{4}$ by 1. By the end of each year, its longer side doubles its length but its other side stays fixed at one-quarter unit.



Students are asked the following parallel questions about the area and perimeter of the figures, with instructions to first predict the answer, then solve the problem, and finally compare their results with their predictions.

Area prompts

1. Compare the areas of the rectangles through the years. Which rectangle is “catching up” with the others and which one “lags behind”?
2. When does the area of each rectangle reach a size of 1,000 square units?

Perimeter prompts

1. Compare the perimeters of the rectangles through the years. Which rectangle is “catching up” with the others and which one “lags behind”?
2. When does the perimeter of each rectangle reach a size of 100 length units?

By design, students are likely to predict, based on visual inspection, that the areas and perimeters of the first and second rectangles remain larger over time than those of the third rectangle. After modeling the situation with a combination of tables, graphs, and functions, however, students will find that the areas and perimeters of the three growing rectangles can be formulated with linear, quadratic, and exponential functions, respectively, and thus that the relatively smaller initial areas and perimeters of the third set of rectangles actually get bigger than those of the first two sets of rectangles, which exhibit linear and quadratic growth. This finding is particularly striking in the tables and graphs, adding useful underlying imagery to the symbolic functions. In addition to highlighting the big idea of patterns of change for linear, quadratic, and exponential functions, this activity gives students an opportunity to hone their modeling competencies by making predictions; analyzing contexts using tables, graphs, and functions; and interpreting their findings. This activity is also an example of how reverse thinking (in this case, determining what input corresponds to a given output) can foster a complementary correspondence view of function. Thus, this task not only engages students in identifying, interpreting, creating, and using nonlinear equations but also helps promote coherent covariation and correspondence views of function.

The growing rectangles task highlights how exponential functions differ from linear and quadratic ones and how the former can be used to model figural growth. As students continue their study of advanced math, they will use exponential functions to model a variety of problems in science, social science, and career and other real-life situations. Examples of such contexts that can be modeled with exponential functions include radioactive decay, the growth of bread mold, the growth of microbes in a pathology test, population growth, compound interest, the outbreak of disease, the growth of invasive plant species, the spread of cancer cells, and even the area damaged in a fire in terms of the duration of burning. Such authentic contexts provide ample opportunities for students to use data, create graphs and equations, and make connections across representations.

(3) MAKING CONNECTIONS ACROSS REPRESENTATIONS

The concept of function and its multiple representations appears across the math curriculum in algebra, calculus, geometry and measurement, and probability and data analysis. Thus, making connections between algebraic, graphical, and tabular representations of all types of functions, of systems of equations, and of inequalities adds power and depth to student understanding. Cooney, Beckmann, and Lloyd (2010, 10), in their influential book *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9–12*, refer to the multiple representations of functions as one of the big ideas of functions foundational to the entire high school math curriculum. The authors highlight the following four essential understandings related to multiple representations:

- "Functions can be represented in various ways, including through algebraic means (e.g., equations), graphs, word descriptions, and tables.
- "Changing the way that a function is represented (e.g., algebraically, with a graph, in words, or with a table) does not change the function, although different representations highlight different characteristics, and some may show only part of the function.
- "Some representations of a function may be more useful than others, depending on the context.
- "Links between algebraic and graphical representations of functions are especially important in studying relationships and change."

The growing rectangles task in the previous section illustrated the power of and need for multiple representations as ways to understand and

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College and Career Readiness

The College Board National Curriculum Survey Report 2019 offers clear indications that advanced math skills and knowledge are important college and career readiness prerequisites.

The sample of 268 postsecondary math faculty rated the following advanced math skill/knowledge elements assessed on the SAT Suite on a four-point scale, with 4 meaning "very important":

- Understand numbers and number systems, including radicals and exponents and their qualities (3.48)
- Represent contexts using a quadratic equation in two variables (2.56), a polynomial expression in one variable (2.90), a rational expression in one variable (2.72), and an exponential equation in two variables (2.27)
- Interpret variables, constants, and/or terms in a quadratic equation (3.33), exponential equation (2.90), and a rational or a radical equation (2.91)
- Use properties of variables to add, subtract, and multiply polynomials (3.55), divide polynomials (2.86), and factor polynomials (3.32)
- Evaluate a polynomial expression (3.45), a rational or a radical expression (3.12), and an exponential expression (3.04)

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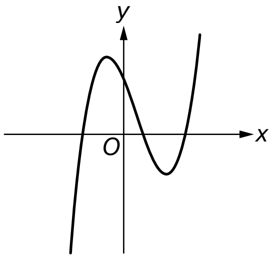
investigate change in area and perimeter, with the tables of values and corresponding graphs highlighting the differences in the rate of change for each function.

Developing flexibility in moving among representations is not something that should be taken for granted, and students need repeated exposure to and practice with these connections (Knuth 2000). At the core of these connections is what is often referred to as the *Cartesian connection*, which states that a point is on the graph of a function if and only if its coordinates satisfy the function (Moschkovich, Schoenfeld, and Arcavi 1993). Calling this connection out as the definition of a graph can help promote meaningful links between the symbolic and graphical representations of function. Earlier in this chapter, we gave the example task of asking students to solve $f(x) = -2$ for x when given a graph of a function, f. This type of problem strongly resonates with the Cartesian connection because one has to make use of the graph to identify the x-coordinate that corresponds to the function value of -2 (which is the y-coordinate value).

Another, slightly more complex task that makes strong use of the Cartesian connection is the graphical connection task described by Friedlander and Arcavi (2017, 126). In this task (figure 3), students are provided with the function $f(x) = (x-1)(x+2)(x-3)$ and asked a series of challenging questions that rely on a sound understanding of the Cartesian connection.



1. $A(2, \_\_\_)$ and $B(-1, \_\_\_)$ are points on the graph of the function. Find their y-coordinates and approximate location on the graph.
2. Find the coordinates of the four intersections of the graph with the axes.
3. Explain why for any $x > 3$ the graph will remain in the first quadrant.
4. Show that the functions $f(x)$ and $g(x) = -5x + 6$ have two intersection points.
5. Choose from the graph an x-interval in which $f(x)$ is negative. Use the algebraic representation of the function to explain why the function is negative in that interval.

Figure 3. Graphical connection task (adapted from Friedlander and Arcavi 2017, 126)

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- Solve a quadratic equation (3.31), a polynomial (degree three or higher) equation in one variable (2.48), an exponential equation in one variable (2.71), a rational or a radical equation in one variable (2.79), and a system of one linear equation and one nonlinear equation (2.28)
- Choose and produce equivalent forms of a quadratic or an exponential equation (2.66)
- Isolate one variable in terms of other variables of an equation (3.59)
- Graph a quadratic equation (3.22), a polynomial (degree three or higher) equation in one variable (2.50), and an exponential equation in one variable (2.63)
- Use and interpret function notation (3.34)

Ratings for the vast majority of these skill/knowledge elements met or exceeded the threshold of 2.50 established as indicating a given skill/knowledge element was considered important as a prerequisite for readiness for postsecondary math.

For more information on College Board's national curriculum survey and its results, see the general introduction to this collection.

Effective responses to these prompts require students to make frequent transitions between the algebraic and graphical representation of the function and to use algebra skills to solve equations. For example, for prompt 4, students may graph both *f* and *g*, conjecture what the two intersection points are, and then use algebra to solve the equation $x^3-2x^2-5x+6=-5x+6$. The equation's structure (in particular, the presence of the expression $-5x+6$ on both sides) offers students a way to simplify the equation, which is then solved with the zero product property (i.e., if $ab=0$, then $a=0$ or $b=0$). As this example illustrates, the tools and techniques of algebra can be used to justify conjectures obtained through graphical analysis.

(4) ANALYZING FAMILIES OF FUNCTIONS

A *family of functions* is a set of functions whose equations have a similar form. The "parent" of the family is the equation with the simplest form. For example, $f(x)=x^2$ is a parent to other functions such as $g(x)=x^2+3$ or $h(x)=(x-4)^2$. Becoming fluent with families of functions entails understanding how (and why) "offspring" functions such as *g* and *h* relate to the parent function, *f*. Students would typically be taught that the graph of the function *g* is a vertical shift upward three units of the graph of *f* and that the graph of *h* is a horizontal shift of *f* four units to the right. The relationships between the graphs of functions *f*, *g*, and *h* are not unique to these functions but apply to all functions. Students can memorize these facts but often have difficulty explaining why these relationships are true in general for all functions. This is especially so for a horizontal translation, which is often counterintuitive (Zazkis, Liljedahl, and Gadowsky 2003). One way to explain why the graph of *h* is shifted four units to the right (instead of four units to the left, as the -4 might suggest) is to leverage the correspondence view of function. For example, knowing that the vertex of the graph of *f* is at $(0, 0)$, one might reflect on what input for *h* is necessary so that the vertex of the graph of *h* has the same output value as the vertex of *f*. The same can be done for all other corresponding input/output pairings for *h* and *f*. A correspondence perspective also helps explain why the graph of *g* is a three-unit shift up of the graph of *f* when one reflects on what, for the same input values, the relationship is between the respective output values. The difference in focus here is that for the horizontal shift one attends to the output invariance, while for the vertical shift one attends to the input invariance.

In addition to horizontal and vertical translations, students should become familiar with a wide variety of transformations and their generalized symbolic formulations as they apply to different families of functions (Cooney, Beckmann, and Lloyd 2010; Leinhardt, Zaslavsky, and Stein 1990; NGA Center for Best Practices and Council of Chief State School Officers 2010). This includes reflection over the *x*-axis $(-f(x))$, reflection over the *y*-axis $(f(-x))$, vertical stretch $(af(x)$ for $|a|>1)$, vertical compression ($af(x)$ for $0<|a|<1$), horizontal stretch ($f(bx)$ for $0<|b|<1$), and horizontal compression ($f(bx)$ for $|b|>1$). The different families of functions that students will encounter in their study of advanced math include the squaring function, $f(x)=x^2$ (which is useful in modeling falling objects); the cubing function, $f(x)=x^3$ (which relates to volume for many physical applications); the square root function, $f(x)=\sqrt{x}$ (which gives rise to the possibility of complex numbers); the reciprocal or rational function, $f(x)=\frac{1}{x}$ (which highlights an issue with the domain and which relates to simple electrical circuits); exponential functions $f(x)=e^x$ or, more generally, $f(x)=a^x$ (which is used in many biology and finance applications); and the absolute value function, $f(x)=|x|$ (which is one of students' first encounters with a function that is not entirely smooth). Understanding the various families of functions just described can help build knowledge toward the logarithmic function $f(x)=log_b x$ (which is closely related to the exponential function and is the basis for the Richter scale) and periodic functions (e.g., $f(x)=\sin(x)$, which is important for modeling tides and other periodic phenomena). These latter two types of functions are prevalent in both math and science classes and offer opportunities to connect science and math (e.g., logarithms and pH).

Function transformations can also be used in novel ways. For example, imagine that, as shown in figure 4, students are given the graph of $f(x)=x-4$ and asked to explain how to use this graph to obtain the graph of $g(x)=|x-4|$. A challenging follow-up task would be to give the split rule definition for *g* and explain how this is related to how the graph of *f* was manipulated to obtain the graph of *g* (adapted from Friedlander and Arcavi 2017, 155).

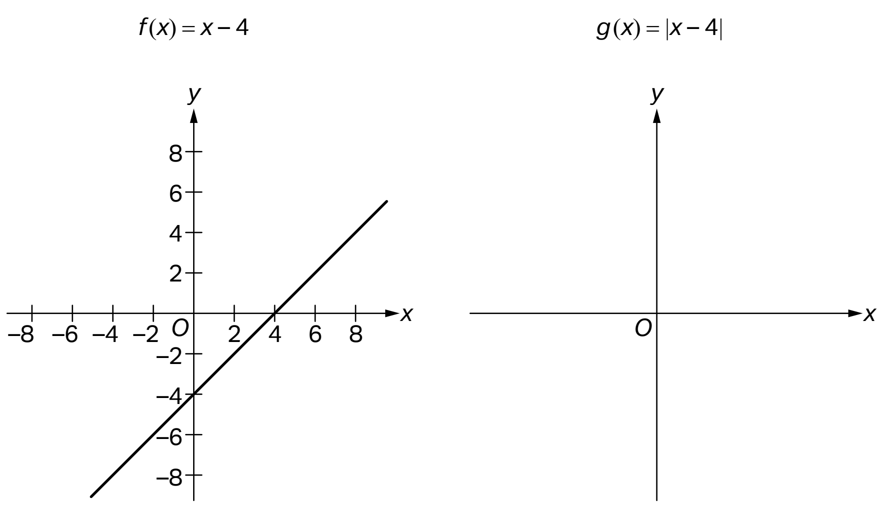

Figure 4. Creating an absolute value function graph (adapted from Friedlander and Arcavi 2017, 155)

Thus far, the discussion of families of functions has focused on the relationship between two different graphs: the parent function graph and an offspring function graph. Building on this pairwise relationship, students should become adept at using parameters in a more global, dynamic way to perform function transformations and investigate properties of families of functions (Cooney, Beckmann, and Lloyd 2010; Hollebrands, McCulloch, and Okumus 2021; NGA Center for Best Practices and Council of Chief State School Officers 2010). This then raises the need to unpack what a parameter is. For investigating families of functions, it is useful to think of a *parameter* as a dynamic, continuously changing quantity that takes on a range of values, each one of which results in a change to the function being examined. When one works with functions in this manner, there is a subtle shift in perspective from thinking about functions as a process of inputs and outputs to thinking about a function as an object (as mentioned previously) in and of itself that can be acted on and changed. Here, a covariation perspective is useful, but, rather than the input and output covarying, the value of n and the resulting graph of f covary.

Consider, for example, the quadratic function $f(x)=x^2+nx+2$ discussed previously. In this case, n would be considered a parameter because as one dynamically varies the value of n, the position of the graph of f dynamically changes. As noted previously, students can graphically explore properties of f. For example, students can use a graph to find the values of n for which the function has two real roots, only one real root, and no real roots. Students can then use algebra to make their graphical investigation precise. Similar graphical-algebraic explorations can be done with parameters a, b, or c in the generalized quadratic function $f(x)=ax^2+bx+c$ as well as with parameters in other families of functions.

4. Implementation Advice

The nature of the math tasks that students encounter is, of course, an important component of powerful and meaningful learning. Students need opportunities to work on challenging, authentic problems, ones that deepen their conceptual understandings, hone their procedural fluency skills, and apply math in real-life contexts (Pimentel 2013).

But what is an authentic problem with corresponding authentic mathematical activity? The draft California mathematics framework (California Department of Education 2022, 24) defines an *authentic activity* or *problem* as “one in which students investigate or struggle with situations or questions about which they actually wonder” and urges that “lesson design . . . be built to elicit that wondering.” As illustrated in some of the sample tasks in this chapter, one way to elicit wondering is to invite students to use dynamic geometry software to explore relationships and properties and make conjectures and then to use algebra to verify or revise their conjectures. Another instructional design strategy that we illustrated in this chapter is reverse thinking. Tasks that require reverse thinking can readily be created from standard textbook tasks. For example, if students are used to being given some bit of information—let's call it X—and asked to find a different bit of information, Y, reformulating this task is simply a matter of asking students what X is necessary to get Y, with perhaps the introduction of a parameter. For example, a standard task is to ask students what the roots are for a given quadratic equation. One can turn this around by inserting a parameter into the function definition and then asking students what parameter value is needed so that the function has two specified roots.

Having good math tasks for students to work on is a necessary but not sufficient condition for meaningful, robust learning. Students also need to be actively engaged in questioning, struggling, reasoning, communicating, making connections, problem-solving, and explaining. Research has repeatedly shown that when students are actively engaged in open-ended, challenging tasks in an intellectually supportive environment, they develop strong conceptual understanding of the math as well as positive identities as powerful doers and learners of math. For example, Boaler (1998) conducted a three-year study of two schools, one using a traditional, textbook-based approach to teaching math and the other using open-ended activities. She found that students in the more traditional classroom developed procedural understandings that were of limited use in unfamiliar situations. By contrast, students who learned math in an open, project-based environment developed conceptual understandings that they could use flexibly and developed habits of mind that helped them in both school and nonschool settings. Such engaged classroom learning environments better prepare students for careers that require interpersonal skills and for subsequent math at both the secondary and postsecondary levels (Association of American Colleges and Universities 2007; Conley 2007b; King et al. 2017; Prinsley and Baranyai 2015).

While K–12 teachers have long recognized the value and benefits of active learning using challenging and authentic math, the postsecondary level is rapidly catching up. For example, the Mathematical Association of America's (2017) *Instructional Practices Guide* provides concrete strategies for classroom, assessment, and instructional design practices as they pertain to active learning. Indeed, the evidence for the benefits of classrooms that actively engage students is now quite strong at the college level. Of particular note, Freeman and colleagues (2014) examined 225 studies that compared student achievement in a range of undergraduate STEM courses and found that students in traditional, lecture-oriented classes were 1.5 times more likely to fail than students in active-learning classes. This and other studies are now resulting in more and more college precalculus and calculus instructors taking up active-learning approaches (Smith et al. 2021). Thus, creating active learning environments at the secondary school level not only helps develop sound and robust learning but also prepares students for the kind of instruction they are increasingly likely to experience at the college level. ❖

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CHAPTER 3

Problem-Solving and Data Analysis

By Anna Bargagliotti

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1. Introduction

Data are everywhere, and working with, understanding, and learning from data have become necessities in our daily lives. Personal data are commonly collected through our digital devices, and our daily behaviors are routinely recorded. Businesses, governments, and other entities use data analytics and powerful computing technology applied to massive pools of information to inform decision-making (Pence 2014), with examples including developing marketing targeted to consumer interests, predicting rises and falls of demand for products and services, improving app-based navigation, aiding healthcare providers in suggesting courses of treatment, detecting financial fraud, and tracking the spread of foodborne illness (Rice 2022; Helms 2015).

Now more than ever, then, it is essential that all students leave secondary school prepared to live and work in a data-driven world (Engel 2017). The development of statistical thinking and data acumen is imperative today, as every individual must use data to make informed decisions involving numerous aspects of their lives (National Academies of Sciences, Engineering, and Medicine 2018; Wilkerson 2020). Many college majors require coursework in statistics (American Statistical Association, n.d.), and statistician jobs are expected to grow by about 35 percent between 2020 and 2030 (U.S. Bureau of Labor Statistics 2021). Postsecondary education in statistics is also changing to meet the demands of twenty-first-century life and careers, with the *Guidelines for Assessment and Instruction in Statistical Education (GAISE) College Report* calling for the preparation of students in statistics at the college level to shift from centering on the application of a list of formulas to a focus on developing the skills of interpretation and understanding data (GAISE College Report ASA Revision Committee 2016). Enrollment of college students in statistics has steadily increased, with the latest (2015) data from the ongoing survey conducted by the Conference Board of the Mathematical Sciences (CBMS) of math and statistics departments at two- and four-year colleges and universities showing that 737,000 students took statistics courses as part of their undergraduate work (Blair, Kirkman, and Maxwell 2018). This represents a 56 percent increase in enrollment in statistics classes since the year 2000, the initial year the CBMS survey was administered.

With ubiquity of data comes responsibility. Although not all students will become statisticians or professional data analysts, they still must be able to check data sources and "mind" the data they encounter. Data minding (Meng 2021, 1161) is a "stringent quality inspection process that scrutinizes data conceptualization, data preprocessing, data curation and data provenance." In other words, students, regardless of their educational plans and intended career paths, must be data literate, able and disposed to act as knowledgeable users of data themselves as well as informed consumers of other people's efforts to use data to support claims and guide actions.

Given the importance of data literacy for college and career readiness, the development of statistics and data literacy should begin early in a person's education (Martinez and LaLonde 2020). Research and recent national reports provide guidance for the ideal case for statistics education, and high-stakes assessment can help support best practices in the classroom. The goal of this chapter is to encapsulate the spirit of best practices for statistical and data literacy in K–12 education.

To guide the development of statistical literacy and statistical thinking, ASA and NCTM commissioned and published the *Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II)* report (Bargagliotti et al. 2020). The report put forth a framework for the teaching and learning of statistics and data science in pre-K–12. The authors anchor the framework within the Statistical Problem-Solving Process, which is defined by four components: formulating statistical investigative questions, collecting/considering data, analyzing data, and interpreting results.[1] As students develop statistical thinking, they move through the process.

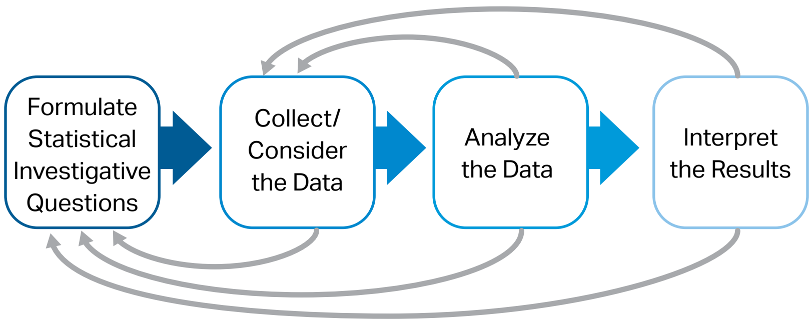

Figure 1. The Statistical Problem-Solving Process (adapted from Bargagliotti et al. 2020, 13)

Several practices provide important pillars for building statistical reasoning abilities and data literacy throughout students' educational careers. Such pillars can be developed through the implementation of the Statistical Problem-Solving Process. In this chapter, we will focus on three such practices: asking questions in statistics, thinking multivariately and using multiple variables, and connecting probability to statistics.

2. Important Statistical Practices

2.1. Asking Questions in Statistics. An important practice that emerges from the Statistical Problem-Solving Process is that of using questioning as a guide through a data investigation (Arnold and Franklin 2021). Asking questions is important, as it is authentic to doing statistics: statisticians are guided by questions about and inquiry into the variability present in data and use questions to frame problems, investigate and collect data, and sift through data to come to purposeful and thoughtful conclusions. As Bargagliotti and Franklin (2021, 15) observe:

> [T]here are questions that motivate a study [research questions], questions that motivate the need to collect data (investigative questions), questions that produce data, questions that prompt analyses of the data, questions that are focused on the interpretation of results, and interrogative questions that are asked as checks and balances throughout the whole process. In essence, questioning can be used throughout the components to guide the investigation and offer insights into each step of the process.

[1] The *GAISE II* framework is based on prior work discussing the teaching and learning of statistics as a process or cycle (Konold and Higgins 2003; Graham 2006; Zayac 1991; Bishop and Talbot 2001). Similar frameworks were put forth by Wild and Pfannkuch (1999), who discussed the investigative cycle in five components—problem, plan, data, analysis, and conclusions (PPDAC)—and the original *Pre-K–12 GAISE* report (Franklin et al. 2005).

As we delve into these statistical practices, it is worthwhile to note some foundational math skills that are necessary to work with data successfully, such as ratios, rates, proportions, and percentages. When students are presented with data or descriptive statistics, the information can consist of the comparison of one given value with another, or how two or more quantities are related. These comparisons could show that the values in two sets of data are related by a common ratio or rate or that one value represents a percent decrease from another. Results of data analysis can include percentages that need to be interpreted to determine the strength of the statistical test used. Essential for students to wrestle successfully with statistical reasoning is their mastery of the concepts of ratio, rate, proportion, and percent.

Although these concepts are typically introduced in the middle school/junior high school math curriculum, their use permeates high school math coursework and beyond. “Facility with proportionality involves much more than setting two ratios equal and solving for a missing term. It involves recognizing quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship” (NCTM 2000, 217). Students need to understand ratio, rate, and proportion before they can experience success in trigonometry and calculus, but these concepts also have a large presence in the sciences, such as when students need to describe speed, acceleration, density, and electric or magnetic field strength. And while a mastery of percentages is necessary for an understanding of exponential growth and decay functions, it is also useful in a variety of real-world situations, including calculating tips, taxes, and discounts (Common Core Standards Writing Team 2011).

In both STEM (science, technology, engineering, and math) and non-STEM fields, ratios, rates, proportions, and percentages can be used to drive analysis and decision-making. These math concepts can help students with data analysis in all fields, shaping their future career experiences.

2.1.1. Questioning within the Formulate Statistical Investigative Questions Component. A statistical investigative question formulated to answer an overarching research question or address a research problem is posed to begin the Statistical Problem-Solving Process. Asking a good statistical investigative question at the outset is important because it sets the stage for a productive statistical investigation. A good statistical investigative question should specify and clarify the variables(s) of interest; the group or population being investigated; the intent of the question; whether the question can be answered with the data collected or whether a study design is possible to collect data to answer the question; and whether the question is interesting, purposeful, and worthwhile to investigate (Arnold and Pfannkuch 2018; Bargagliotti and Franklin 2021).

Table 1, adapted from the *GAISE II* framework for levels A, B, and C (roughly corresponding to elementary, middle, and high school levels, respectively, in the United States), illustrates the types of questions that can be posed to begin a statistical investigation at each level.

Process Component	**Level A**	**Level B**	**Level C**
I. Formulate Statistical Investigative Questions	Understand when a statistical investigation is appropriate Pose statistical investigative questions of interest to students where the context is such that students can collect or have access to all required data Pose summary (or descriptive) statistical investigative questions about one variable regarding small, well-defined groups (e.g., subset of a classroom, classroom, school, town) and extend these to include comparison and association statistical investigative questions between variables Experience different types of questions in statistics: those used to frame an investigation, those used to collect data, and those used to guide analysis and interpretation	Recognize that statistical investigative questions can be used to articulate research topics and that multiple statistical investigative questions can be asked about any research topic Understand that statistical investigative questions take into account context as well as variability present in data Pose summary, comparative, and association statistical investigative questions about a broader population using samples taken from the population Pose statistical investigative questions that require looking at a variable over time Understand that there are different types of questions in statistics: those used to frame an investigation, those used to collect data, and those used to guide analysis and interpretation Pose statistical investigative questions for data collected from online sources and websites, smartphones, fitness devices, sensors, and other modern devices	Formulate multivariable statistical investigative questions and determine how data can be collected and analyzed to provide an answer Pose summary, comparative, and association statistical investigative questions for surveys, observational studies, and experiments using primary or secondary data Pose inferential statistical investigative questions regarding causality and prediction

Table 1. Formulate Statistical Investigative Questions component (adapted from Bargagliotti et al. 2020, 16)

Among other features, the table shows the increasing sophistication across grade levels of statistical investigative questions, which shift from descriptive (e.g., What types of music are preferred by middle school students?) to comparative (e.g., How do sleep patterns of students in different grades compare?) to predictive (e.g., How can the mass of a lizard be used to predict its habitat location?) and inferential (e.g., What are plausible ages for pennies in circulation?).

2.1.2. Questioning within the Collect/Consider the Data Component. Regardless of whether the data to be studied are primary or secondary, we need to interrogate the data collection methodology. With primary data—data collected by the researcher—we might use questions specifically within the data collection process to conduct a survey or collect data in some other way. We also use questioning to interrogate how to set up the study design (e.g., How does the study design relate to the statistical investigative question posed? Are the observations representative of the population?).

When using secondary data—data collected by someone else—we must understand the provenance of data by asking questions about the manner in which the data were generated. Here, we do not collect the data ourselves but instead use questioning to interrogate the data (e.g., How were the data collected? What are the units of observation? Are the data appropriate to answer the statistical investigative question?).

Whether the data are primary or secondary, interrogation to understand the data collection methodologies or clarify the study design is imperative to working through the statistical process (Arnold et al. 2022). Furthermore, students should ask questions of the dataset to help them understand its structure (e.g., What variables are defined? How many observations are there?). Important issues to consider—and matters that can be explicitly explored through questioning within this component, particularly when working with secondary data—are whether the dataset might have bias and whether the dataset is appropriate to answer the formulated question. (See Rodham and Gavin 2006; Richards and King 2014; Wainwright and Sambrook 2010; Saltz et al. 2019; Rhode 1992; Martin 2015; and American Statistical Association 2022 for provoking thoughts on the ethics of working with data.)

2.1.3. Questioning within the Analyze the Data Component. When moving to the analysis phase of the Statistical Problem-Solving Process, a researcher can pose directed questions that lead to specific analyses being performed: What is the mean of a variable of interest? What is the frequency of observations for different categories of a variable? How is a variable distributed? How much do the data vary in the distribution of the variable? Can the data be visualized in a scatterplot? Are the two quantitative variables being studied correlated? Such questions are examples of how guided analysis through questioning can take place. It is important for students to remember that analysis questions should be posed with the goal of helping them answer the statistical investigative question.

2.1.4. Questioning within the Interpret the Results Component. Lastly, in the Interpret the Results component, questioning is used to summarize and regroup all the information uncovered by the analysis into a cohesive answer: What do the analyses tell us? How can the analyses be incorporated to provide an answer to the investigative question? Do our data answer the investigative question, or do more and/or different data need to be collected?

2.1.5. Example Investigation. The following example demonstrates the practice of using questioning throughout the statistical investigative process.

Example 8 for level C in the *GAISE II* report (Bargagliotti et al. 2020) classifies lizards using prediction. To model how questioning plays an important role throughout a complex investigation such as this, we begin by posing a statistical investigative question: How can a lizard's mass be used to predict whether it came from a "natural" or a "disturbed" habitat?

A biologist[2] randomly captured a total of 160 *Anolis sagrei* (brown anole) lizards across two types of habitats on four islands in the Bahamas and recorded various measurements for each lizard. Of the 160 lizards, eighty-one were from a natural habitat, and seventy-nine were from a disturbed habitat. A *disturbed habitat* was defined as a habitat where there was a human presence, whereas a *natural habitat* was one where there was no human influence.[3]

Questioning and interrogating the secondary data reveal there are twelve variables in the dataset, of which two are categorical variables and ten are quantitative measurement variables. Based on preliminary studies, the biologist believed that mass (in grams [g]) would be a good predictor of habitat. The variables of current interest were therefore "habitat" and "mass (g)." "Habitat" is a two-category variable, while "mass (g)" is quantitative. Figure 2, produced in the Common Online Data Analysis Platform (CODAP), shows the distribution of mass by habitat category.

An important analysis question is, How can the two distributions of mass by habitat be described and compared through their shapes, centers, variabilities, and unusual features as well as in terms of separation versus overlap? We observe from the figure that the center of the disturbed-habitat distribution is shifted to the right of the natural-habitat distribution

[2] The data used in this investigation were collected by Erin Marnocha during her time as a graduate student at the University of California, Los Angeles.

[3] The data collected can be accessed in the Common Online Data Analysis Platform (CODAP): **https://codap.concord.org/app/static/dg/en/cert/index.html#shared=https%3A%2F%2Fcfm-shared.concord.org%2FQEa7RiiLSv57fZSnXJR1%2Ffile.json**.

and that the disturbed-habitat distribution shows more variability than the natural-habitat distribution.

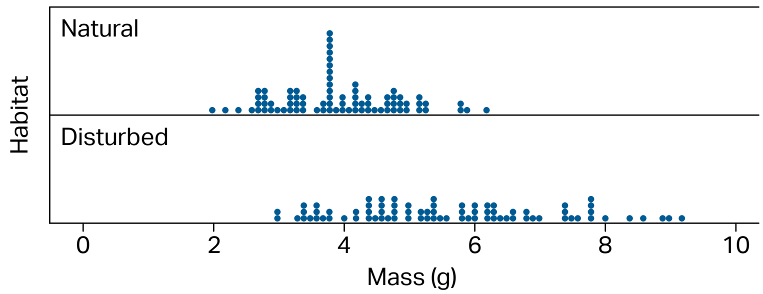

Figure 2. Distributions of lizard mass by habitat (adapted from Bargagliotti et al. 2020, 99)

We also observe that predicting a lizard's habitat is not straightforward because of the large overlap in the distributions in figure 2. One cannot draw a single vertical line to cleanly divide the observations plotted on each dot plot into two groups (left of the line and right of the line), which would create a classification rule that would correctly classify *all* lizards. Instead, students can explore how to place a cutoff line and the effects of its placement on prediction accuracy. Because of a separation of the two distributions, a plausible vertical cutoff line could be drawn at 6.25 g (as shown in figure 3) such that all the lizards that were collected from a natural habitat would be classified correctly. However, several lizards from a disturbed habitat would be classified incorrectly.

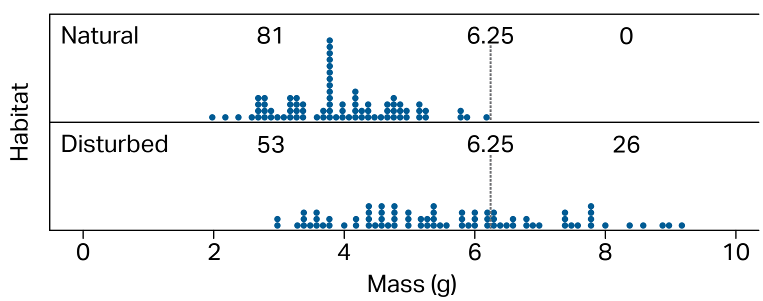

Figure 3. Distribution of lizard mass by habitat with cutoff line (adapted from Bargagliotti et al. 2020, 99)

A confusion matrix for this rule is given in table 2. The 6.25 g cutoff rule results in a misclassification rate of 33 percent, calculated as the number of lizards misclassified divided by the total number of lizards: $\frac{53}{(81+53+26)}$.

	Classified as "natural"	Classified as "disturbed"
Truly from a natural habitat	81	0
Truly from a disturbed habitat	53	26

Table 2. Confusion matrix for 6.25 g cutoff line (adapted from Bargagliotti et al. 2020, 99)

Students can be challenged to improve on this misclassification rate by testing other potential vertical cutoffs. This example is further expanded on in the *GAISE II* report to show how multiple variables can be used as potential classifiers to help predict the habitat, with questioning again guiding the analysis. For example, mass and head depth could both be used to predict habitat. Analysis questions such as the following could be used as guides to whether linking mass and head depth would be productive: Is there a linear relationship between mass and head depth? Is the relationship positive or negative? What is the strength of the relationship? What is a possible fitted line to the data?

Students might also construct prediction intervals and confidence intervals in this analysis while they consider the categorical variable of habitat. What would happen if two separate lines were fit? Would the prediction be improved? These types of analysis questions can deepen the investigation. In the example CODAP output in figure 4, we see that the "natural" (orange) dots appear to be clustering together and the "disturbed" (blue) dots appear to be clustering together. The regression lines by category are estimated as follows and pictured in figure 4:

Predicted mass for the disturbed-habitat lizards $= 2.13 \times$ head depth $- 5.4$

Predicted mass for the natural-habitat lizards $= 1.415 \times$ head depth $- 3.1$

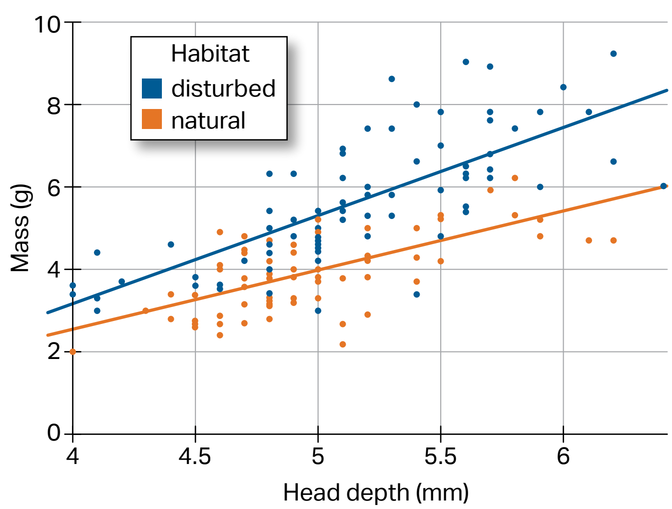

Figure 4. Scatterplot with regression lines for lizard mass and head depth in disturbed and natural habitats (adapted from Bargagliotti et al. 2020, 99)

In statistics, both classification and regression are used to make predictions; however, differences arise between the two methodologies based on the type of dependent variable one is trying to predict. In figure 4, we illustrate using bivariate regression analysis while incorporating a third categorical variable for classification purposes. There are, in fact, different entry points for students to develop a prediction using a sound statistical justification. They could plot the point on the scatterplot and either observe which cluster the point better fits or which line the point falls closer to, or they could evaluate the two equations of the lines and consider the residuals (distances that measure the variability around the fitted line).

Questioning occurs and serves essential roles throughout the Statistical Problem-Solving Process. Within the Formulate Statistical Investigative Questions component, posing a well-constructed question allowed us to focus the investigation. In the Collect/Consider the Data component, questioning helped us understand the data and whether the secondary data were appropriate to analyze for the posed investigative question. In the Analyze the Data component, questioning guided the types of calculations and graphical displays we produced. Finally, in the Interpret the Results component, questioning helped tie all the analyses together to answer the original statistical investigative question.

2.2. Thinking Multivariately and Using Multiple Variables. The onslaught of data has brought about a plethora of rich datasets that contain many different types of variables. Today's variables are not only categorical and quantitative but also pictures, sounds, video, and words (Gould 2010). *GAISE II* (Bargagliotti et al. 2020, 9) advises that "students need to be able to identify raw data of these non-traditional variable types, understand how variable transformations can produce different representations of the same data, and organize these data appropriately." As the types of variables and the size of datasets (in terms of both number of variables and number of observations) expand, students must grapple with how to think about multiple variables at a time. Complex problems require us to consider multiple variables in order to fully understand a situation. Without considering multiple variables and thinking multivariately, students may arrive at overly simplified and possibly flawed understandings of the situation that they are investigating. We illustrate the importance of considering multiple variables in two short examples.

2.2.1. Example Investigation 1. Complex data visualizations showing relationships among multiple variables are a common feature in news media. Such data visualizations can provide a good starting point for students to consider multiple variables.

"What's Going On in This Graph?" (WGOITG), an ongoing weekly collaboration between the *New York Times* and ASA, presents data visualizations that previously appeared in the *Times* and hosts ASA-moderated live discussions of those visualizations for students. The September 15, 2021, WGOITG post (**https://www.nytimes.com/2021/09/09/learning/whats-going-on-in-this-graph-sept-15-2021.html**), for example, used data from the U.S. Bureau of Labor Statistics' American Time Use Survey (ATUS) (**https://www.bls.gov/tus/**) to address the question of how the coronavirus pandemic changed how people in

This chapter builds an argument that it is essential that students leave secondary school prepared to work with data, armed with statistical thinking skills and data acumen. Additionally, students need to understand concepts from the study of probability in order to understand the importance of randomness in statistics.

Two foundational topics that flow through the math curriculum, typically starting in grade 6 and continuing through high school, are developing an understanding of proportional reasoning and applying proportional relationships to solve single-step and multistep problems. Proportional reasoning is an important skill when solving percent-based problems, such as those involving discounts, tips, sales tax, interest, unit rates, and percent increase and decrease, and thus it is assessed, at appropriately challenging levels, throughout the digital SAT Suite, including on the SAT.

Unlike topics covered in the Algebra and Advanced Math content domains, the topics addressed by the digital SAT Suite in Problem-Solving and Data Analysis are not aligned to those covered in a specific secondary-level course. State education systems include the topics covered in the Problem-Solving and Data Analysis domain in a variety of courses, starting with those in middle school/junior high school and continuing through those in high school. The test questions in this domain range in difficulty from relatively easy to relatively complex and challenging and test a wide range of reasoning skills. Problem-Solving and Data Analysis questions address the following skill/knowledge elements in proportions and ways varying by testing program, as shown in table 3, below.

	Digital SAT Suite testing program (percent of section)		
Skill/knowledge element	**SAT (15%)**	**PSAT/NMSQT and PSAT 10 (20%)**	**PSAT 8/9 (25%)**
Ratios, rates, proportional relationships, and units	all	all	all
Percentages	all	all	all
One-variable data: distributions and measures of center and spread	all	all	analyze data representations, measures of center, compare data sets, outliers
Two-variable data: models and scatterplots	all	all	analyze data represented in a scatterplot, fit linear models, read and interpret graphs modeling data
Probability and conditional probability	all	all	all
Inference from sample statistics and margin of error	all	sample means	not covered
Evaluating statistical claims: observational studies and experiments	all	not covered	not covered

Table 3: Topics addressed in the Problem-Solving and Data Analysis math content domain, by testing program

the United States spent their free time. This particular graphic shows the amount of time people in different age brackets (15–24, 25–44, 45–64, 65+) spent per day in 2019 and 2020 on texting, phone calls, and video chats; exercising; and grooming. Five different variables are used in the graphic: age group; year; time spent on texting, phone calls, and video chats; time spent exercising; and time spent grooming. Arnold et al. (2022) discuss the connections of this WGOITG post to the *GAISE II* level C framework and have curated the full ATUS dataset used by the *Times* as well as a sample of four thousand observations from the ATUS data for classroom use.

Using the sample curated dataset, students can further investigate ATUS to pose and answer other multivariate statistical investigative questions. For example, a student might ask whether gender differences exist across age categories in the amount of time people sleep. Figure 5 shows that while people of all age categories appear to sleep similar amounts on average per week, the distribution shapes vary across categories. The sleep amount of people in the age category 15–24 has an approximately uniform distribution, indicating wide variability among this group in terms of the amount they sleep. Sleep amounts of people in the age categories of 45–64 and 65+ have similarly shaped distributions, with distinct peaks around 550 minutes. There do not appear to be any gender differences across the age categories, which can be seen by the similar distributions of blue (female) and orange (male) dots.

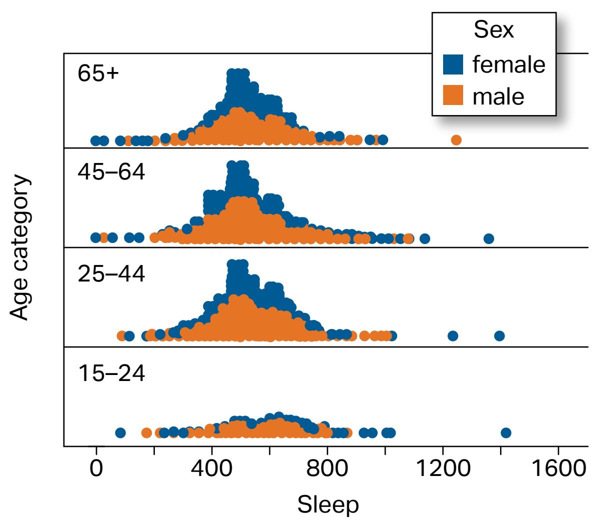

Figure 5. Minutes spent sleeping per day by age category and gender (source: Arnold et al. [2022] curated ATUS sample of four thousand observations)

Similar questions highlighting the importance of simultaneously considering multiple variables can be explored using the sample curated ATUS data. Students can, for instance, be encouraged to look at the associations between two quantitative variables using regression while introducing a third categorical variable (similar to the lizards example above) or to examine the association between two categorical variables through contingency tables, relative ratios, and percentages. Furthermore, the ATUS sample data can be used to draw inferences about the full ATUS dataset, which contains over six hundred thousand observations. Students can estimate relationships between quantitative variables and estimate prediction intervals as well as confidence intervals by computing the appropriate margins of error for each.

2.2.2. Example Investigation 2. Given the proliferation of mobile devices with high-resolution cameras, photographs have increasingly become a way to document our lives. Arnold, Johnson, and Perez (2021) discuss the possibility of using such photographs as data sources in the classroom. Johnson (2021) details a lesson in which students identify their favorite outdoor space, take a picture of it, and submit the picture along with information about the space to a class dataset. Example variables that can be explored via the resultant dataset include how students get to these spaces (car, bike, skateboard, walk/run), how often in the last seven days students enjoyed their spaces, what percentage of the spaces consist of human-made objects, what color is primarily represented in the spaces, and whether students exercise in the spaces. Using a class dataset, students can, for instance, investigate the relationship between the type of transportation used to get to the spaces and the number of days they visited the spaces in the last week. Figure 6 shows four box plots, one for each transportation category, allowing students to compare the distributions of the number of days they visited their respective spaces in a week.

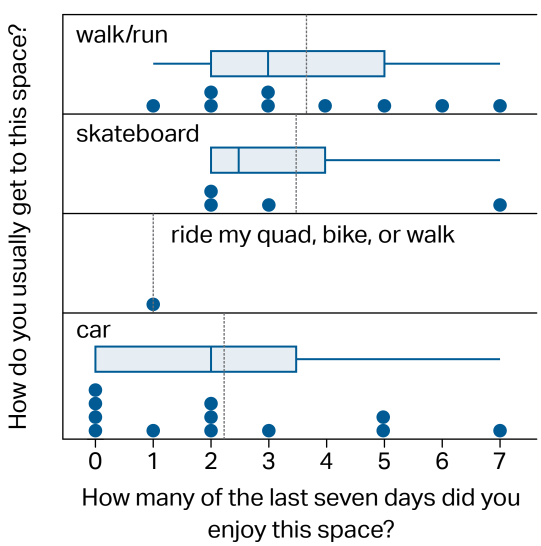

Figure 6. Number of days in the previous week that students visited their favorite outdoor spaces, by mode of transportation (adapted from Johnson [2021], 11)

Figure 6 shows that 46 percent of the students got to their favorite place by car, 35 percent by walking or running, and 15 percent by skateboard. One student responded with "other" and supplied a response of riding a quad (ATV), biking, or walking. The type of transportation appears to be associated with the number of days the favorite place was visited. For example, favorite places that could be accessed by walking or running had a higher mean and median number of days visited per week than those accessible by the other transportation methods. From the box plots, students can compare interquartile ranges to further draw comparisons among the transportation types.

Frequency of visits may also be related to what extent the favorite outdoor space is human-made or natural. Figure 7's dot plot depicts the relationship among these three variables.

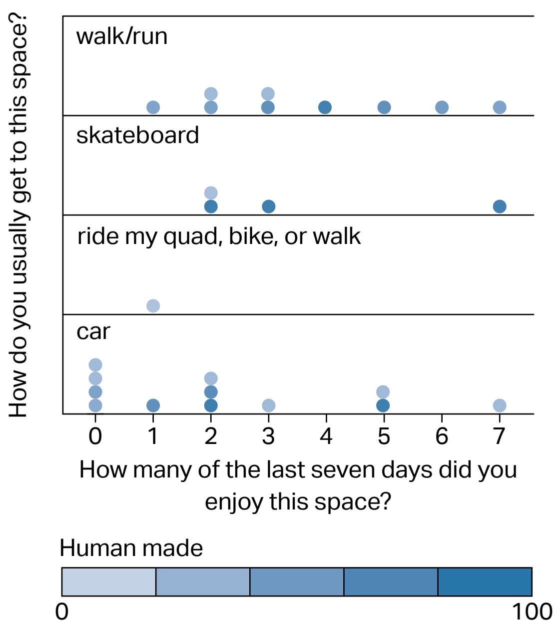

Figure 7. Number of days in the previous week that students visited their favorite outdoor spaces, by transportation type and by the percentage of the space that is human-made (adapted from Johnson [2021], 12)

The scale of blue shading used for the dots in figure 7, identified at the bottom of the figure, represents the percent of each photographed outdoor space that is human-made as opposed to natural: the darker the shade of blue, the higher the percentage of human-made features. Figure 7 shows that over 60 percent of the favorite places had low percentages of human-made features, as represented by dots of the lightest two blue shades. These more natural spaces were often reached by walking/running or biking and were sometimes accessed several days during the seven-day span studied. This information might, by itself, suggest a relationship between transportation type, number of visits, and the extent to which a space was natural. However, students using skateboards as transportation frequently visited spaces that had higher proportions of human-made features. Overall, it appears that the percent of human-made features is not strongly associated with the number of times a space was visited, but it might still be associated with the type of transportation used to access the space.

2.3. Connecting Probability to Statistics. Probability provides the tools to model and quantify randomness. A *probability* is a quantity between zero and one that defines the likelihood of the occurrence of an event. For certain types of random processes (such as tossing a coin or rolling a die), probability represents the long-run relative frequency of an event. In other cases, such as sporting events, probability represents the likelihood that a single event (such as a particular team winning a game) will occur.

Statistics and probability are connected via randomness: statistics aims to draw conclusions in the presence of variability in data. It is important that students develop probabilistic thinking not only to attain a sense of likelihood but also to understand inference and cause and effect. Statistics uses probability through random assignment and random sampling. Random assignment in a study helps balance out the effects of potentially confounding variables and provides the foundation for establishing cause-and-effect relationships between variables. Ideally, random selection produces a sample that is representative of the targeted population so that sampling bias is reduced.

Designing and carrying out simulations are integral parts of developing student understanding of the connection between randomness and probability. As Bargagliotti and Franklin (2021, 142) write:

> The idea of randomness begins with the notion that an individual outcome from a repeatable random process cannot be predicted with certainty. However, if the random process is repeated a large number of times, a predictable pattern in the relative frequency of outcomes generated from this process will emerge. Probability is the branch of mathematics that seeks models for describing this long-run predictable pattern. These models provide order to the seeming disorder present in the outcomes from the individual trials.

Students' development of probabilistic thinking can begin at a young age (Jones et al. 1999; Batanero et al. 2016; Nikiforidou, Pange, and Chadjipadelis 2013). Foundational work by Fischbein (1975) lays out how probabilistic thinking is intuitive in children. Other researchers have advanced the work of Fischbein and others by discussing students' informal knowledge of probability (Greer 2001), developing methods for teaching and learning probability that confront misconceptions (Shaughnessy 1992; Batanero et al. 2016), and discussing ways to

CollegeBoard

College and Career Readiness

The *College Board National Curriculum Survey Report 2019* offers strong indications that problem-solving and data analysis skills and knowledge are important college and career readiness prerequisites.

The sample of 268 postsecondary math faculty rated the following problem-solving and data analysis skill/knowledge elements assessed on the digital SAT Suite on a four-point scale, with 4 meaning "very important":

- Understand numbers and number systems, including absolute value of real numbers (3.38) and elementary number theory (primes, prime factorization, divisibility, number of divisors, odd/even) (3.26)
- Solve problems with rates, ratios, and percents (3.49)
- Use units and unit analysis to solve problems (3.20)
- Identify and distinguish linear and exponential growth (2.74)
- Given a scatterplot, model statistical data with a linear function (2.65), a quadratic function (2.09), and an exponential function (2.03)
- Solve problems using measures of center, including mean, median, and mode (2.60), measures of spread, including range and standard deviation (2.24), sample statistics and population parameters (2.09), and probability (2.18)
- Understand the characteristics of well-designed studies, including the role of randomization in surveys and experiments (2.06)
- Read and interpret statistical graphs (2.48)

(continued on next page)

integrate probability into the math and statistics curriculum (Watson 1995; Ahlgren and Garfield 1991; Carpenter et al. 1981; Jones 2005).

Probability is a tool in statistics for understanding the probabilistic arguments of chance embedded in random assignment and random sampling study designs. To introduce students to the idea of a random process, consider the simple example of flipping a coin. When you flip a coin, you do not know with certainty whether the outcome is going to be heads or tails, nor do you know exactly what proportion of heads will be seen in the first ten flips. However, assuming the coin is fair (that is, each side of the coin has the same chance of turning up after a flip), we *do* know that if we flip a coin many times, in the long run about half the flips should land heads-up and about half should land tails-up. While outcomes are unpredictable in the short run, patterns emerge in the long run.

While probability is important in statistics, it is also a standalone branch of math. As pointed out in the *GAISE* (Franklin et al. 2005) and *GAISE II* (Bargagliotti et al. 2020) reports, important distinctions can be made between two types of probabilistic questions:

> (1) Assume a standard six-sided die is fair. If the die is rolled ten times, how many times will we observe an even number on the top face?
>
> (2) You pick up a standard six-sided die. Is it a fair die? That is, does each face have an equal chance of appearing when the die is rolled?

These two questions highlight how probability can be conceptualized and used. For the first question, probability is used in a mathematical manner and is computed and logically deduced. For the second question, probability is a tool used to find a solution by observing experimental results. This second question highlights the connection between statistics and probability. For example, suppose the die is rolled a thousand times and the number 1 comes up four hundred times. In statistics, we aim to understand whether it is likely this happened by chance (i.e., rolling a 1 on a fair six-sided die 40 percent of the time is a typical result) or whether this is evidence that the die is likely not fair (i.e., the probability of rolling a 1 on a fair six-sided die 40 percent of the time is low).

The report *Catalyzing Change in High School Mathematics* (National Council of Teachers of Mathematics 2018, 61) notes that "studies that are widely reported in the media often contain generalizations about a population or a comparison of experimental treatment groups." Inferences are the basis for moving from analyzing sample statistics to drawing reasonable conclusions about the parameters of the population from which the sample was drawn (Moore 1990). Probability is the reason why such conclusions can be drawn: random selection of a sample allows for valid estimation of population parameters such as a population mean or a population proportion. Students should note that the types of conclusions one can reasonably make from sample statistics are

(continued from previous page)

Ratings for many of these skill/knowledge elements met or exceeded the threshold of 2.50 established as indicating a given skill/knowledge element was considered important as a prerequisite for readiness for postsecondary math. In addition, four of the skill/knowledge elements considered unimportant as college and career readiness prerequisites by postsecondary math faculty were considered important by postsecondary science faculty:

- Solve problems using sample statistics and population parameters (2.56)
- Solve problems using probability (2.65)
- Understand the characteristics of well-designed studies, including the role of randomization in surveys and experiments (2.91)
- Read and interpret statistical graphs (3.24)

Taken together, these results indicate that nearly all the Problem-Solving and Data Analysis elements measured on the SAT Suite are considered important by at least one postsecondary faculty group surveyed.

For more information on College Board's national curriculum survey and its results, see the general introduction to this collection.

determined by the type(s) of data collected and the study design that was used. To make valid inferences from samples to populations, the sample must be representative of the population from which the sample is drawn. The larger the sample size, the less the variability in the distribution of a sample statistic (such as the sample mean or sample proportion) is expected to be and, thus, the greater the precision of the inferences that can be drawn about population parameters.

When studies make causal claims—for example, a particular math teaching intervention increases student achievement—random assignment of study participants to treatment and control groups is required. In this case, the treatment group would experience the math teaching intervention, while the control group would experience traditional teaching. The two groups would vary only with respect to the type of teaching they received; all other factors (e.g., time taught) would be held constant across the two groups to increase the likelihood that any significant difference in achievement between the two groups can safely be attributed to the intervention.

Connections between probability and statistics can be developed through the use of simulations (both manual and technology aided). Simulations can also be used to derive probability formulas. The two short examples below illustrate both the mathematical use of probability and the use of probability as a tool in statistics.

2.3.1. Example Investigation: Mathematical Probability. A popular probability investigation discussed by many (e.g., Tyson and Wilcox 2016; Bargagliotti and Franklin 2021) and also featured in a TED Ed talk (**https://ed.ted.com/lessons/the-last-banana-a-thought-experiment-in-probability-leonardo-barichello#review**) has students conduct the following two-player activity: Two people on a deserted island play a game to determine who will get the last banana to eat. The people have two six-sided dice with them. Each player rolls a single die. If a 1, 2, 3, or 4 is the highest number rolled, player A wins. If, instead, a 5 or a 6 is the highest number rolled, player B wins. Students can predict which player will win and simulate the game to compute an empirical probability. For example, two students simulated the game twenty times, with player A winning nine times and player B winning eleven times.[4] From the simulations, students will observe that player A wins approximately 45 percent of the time and player B wins approximately 55 percent of the time. The probability of each player winning can also be calculated using a theoretical formulaic approach. Since the two die rolls in each game are independent, $P(A\ wins) = P(\text{die 1 has an outcome of 1, 2, 3, or 4}) \times P(\text{die 2 has an outcome of 1, 2, 3, or 4})$. Therefore, $P(A\ wins) = \left(\frac{4}{6}\right) \times \left(\frac{4}{6}\right) = \left(\frac{16}{36}\right) \approx 0.44$, and since $P(A\ wins) + P(B\ wins) = 1$, then $P(B\ wins) = 1 - 0.44 = 0.56$.

[4] Using technology, the game can quickly be simulated many times, permitting the easy accumulation of longer-run results.

This example shows the computation of a mathematical probability that was derived both through a simulation-based approach and a formulaic approach.

2.3.2. Example Investigation: Probability as a Tool in Statistics. To explicitly connect probability to statistics, consider another popular example (**https://askgoodquestions.blog/2019/11/11/19-lincoln-and-mandela-part-1/**) that uses simulations to develop the idea of a sampling distribution for the sample mean using the words in President Lincoln's Gettysburg Address as the population and the mean length of the words in a sample as the statistic of interest (the sample mean). A student can use technology to randomly sample words and determine their mean length. The results of randomly collecting one thousand samples of ten words each and computing each sample's mean word length are displayed in figure 8.

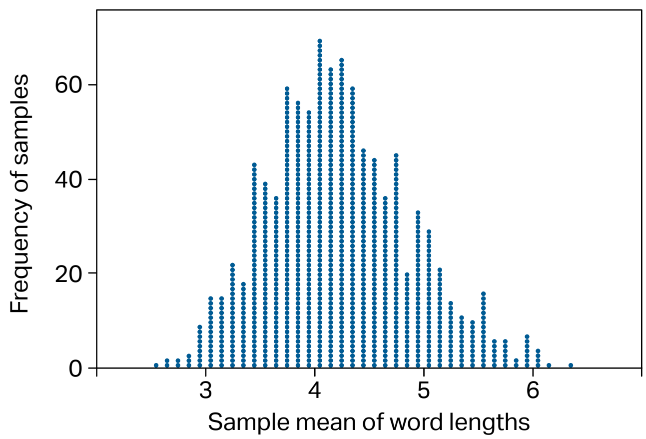

Figure 8. Approximate sampling distribution of mean lengths of one thousand random samples of ten words from the Gettysburg Address (adapted from Bargagliotti and Franklin [2021], 188)

Each dot in figure 8 represents the mean word length for one random sample of ten words. The figure includes one thousand dots because the software used in the analysis selected a thousand different random samples of ten words each. We can observe from the dot plot that there is a great deal of variability in the mean word lengths from sample to sample, with some means being as low as 2.5 letters and some as high as 6.5 letters. Overall, the approximate sampling distribution appears to be bell-shaped and centered on 4.3 letters. This center is the mean of the means, the average of the sample averages. This indicates that in this long-run simulation, the mean of the sample means settles around 4.3. This long-run connection to probability for the mean of the means is the basis for statistical inference connecting samples to populations. Understanding the sampling distribution of a sample statistic (such as the sample mean or a sample proportion) can show us the likelihood of certain values of the sample statistic. Based on these likelihoods (probabilities), we can estimate population parameters, thus connecting probability to inferential statistics.

3. Connections to the Statistical Problem-Solving Process and Conclusions

Teaching statistics requires making sense of context and understanding variability in data. Students need to acknowledge variability present in data and to use models to describe this variability. Statistics students need to carry out the Statistical Problem-Solving Process by using questioning to guide their inquiry and by thinking multivariately and considering multiple variables. Understanding how probability connects to randomness also allows students to grasp the idea of cause and effect and lays the foundation for inference. As the examples in this chapter have illustrated, technology is an important tool in modern statistics; applets and statistical software are necessities for robust data analysis in the classroom. Best practice in the classroom and in assessment should focus on all components of the Statistical Problem-Solving Process: formulating statistical investigative questions, collecting/considering the data, analyzing the data, and interpreting the results. Educators must strive to engage students in this process in order to prepare them for making sense of the wealth of data surrounding us all. ❖

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CHAPTER 4

Geometry and Trigonometry

By Erin E. Krupa

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1. Introduction

Because geometry, historically the study of shapes and their properties, originates in the study of the measurement of the earth (*Merriam-Webster* 2021), it is one of the oldest branches of math and is, in some ways, the most immediately relevant. Freudenthal (1971) argued that the study of math should be tied to the world in which we live, else it is easily forgotten and rarely used. Geometry is inherently related to modeling the world around us, which includes measuring objects in space and developing spatial and deductive reasoning.

The value of geometry in K–12 education extends beyond the typical merits of understanding a subject to helping lay the foundations for achievement in other branches of math. Topics in geometry and measurement were considered Critical Foundations of Algebra by the National Mathematics Advisory Panel (NMAP) (2008). The report the panel produced specifically discussed the importance of similar triangles to understanding slope and linear functions. In addition, the panel suggested that to prepare for algebra, "students should be able to analyze the properties of two- and three-dimensional shapes using formulas to determine perimeter, area, volume, and surface area" and "should also be able to find unknown lengths, angles, and areas" (18). Similarly, the authors of the Common Core State Standards for Mathematics (CCSSM) (NGA Center for Best Practices and Council of Chief State School Officers 2010, 84) observed that "solving real-world and mathematical problems involving angle measure, area, surface area, and volume" are high priorities for college and career readiness. Additionally, a survey of college math faculty (Er 2018) rated reasoning and generalization—skills developable through the study of geometry—as both the most important math competencies for incoming college students to have previously mastered and the least likely to have been attained. Further, the study of geometry prepares students for trigonometry and precalculus, the latter of which Atuahene and Russell (2016) have shown that 53 percent of first-year college students struggle with, earning D, F, or W (withdrawal) grades at the end of a semester-long course. Geometry and trigonometry content is important not only academically for STEM fields (e.g., engineering, medicine) but also for careers in the trades (e.g., transportation, construction) and the arts (Morgan 2018).

Historical assessment data from students in the United States relative to students from other nations show a long-term trend of weak performance on items related to geometric reasoning and measurement (Carpenter et al. 1980; Fey 1984; Stigler, Lee, and Stevenson 1990). More recent findings have not improved the picture. Data from the Trends in International Mathematics and Science Study (TIMSS) highlighted geometry and measurement as the biggest areas of weakness for eighth-grade students from the United States (Ginsburg et al. 2005), and geometry performance by U.S. high school students was the lowest among the sixteen participating countries (Mullis et al. 1998). The most recent National Assessment of Educational Progress (NAEP) with publicly released test items (NAGB 2013) includes a grade 12 question asking students to determine the area of a triangle in a 3D figure. Only 5 percent of U.S. students were able to give a correct answer and show how they found the area of the figure. Additionally, it is well documented that U.S. high school students struggle with formal proof (e.g., Stylianides, Stylianides, and Weber 2017), which is why they need more opportunities for informal reasoning and sense making. These data are concerning given the importance of these topics for college and career readiness.

Recently, there has been a renewed focus on the teaching and learning of geometry in grades 6–12 owing to the relevance of geometry to daily life (McCrone et al. 2010). Middle school is where students begin learning about transformations, similarity, congruence, the Pythagorean theorem, and the measurement of angles, area, surface area, and volume. These ideas are formalized in high school as students use precise definitions for geometric objects, develop proofs, explore geometric transformations to understand triangle similarity and congruence, and are introduced to right triangle and unit circle trigonometry. In addition, applying transformations and investigating symmetry are becoming increasingly important in the geometry curriculum (Sinclair, Pimm, and Skelin 2012a, 2012b). Geometric transformations are well connected to ideas from algebra and trigonometry (coordinate planes, matrices, and vectors) and are very important in emerging technologies in such fields as computer graphics, computer-aided design, and video game development.

It is important to provide students with opportunities to reason and make sense of geometry in order to build their capacity for formal deductive reasoning and to prepare them for college and careers. To build geometric reasoning, students need to engage in iterative cycles of investigation, conjecture, justification, and proof, as illustrated in figure 1.

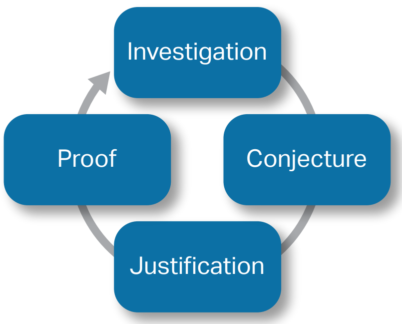


Figure 1. Geometric reasoning cycle

Investigation refers to exploring geometric objects and their relationships and properties. As Sinclair, Pimm, and Skelin (2012b, 25) observe, "Underlying any geometric theorem is an invariance—something that does not change while something else does." During investigations, students explore geometric objects and begin to notice these invariances, which highlight important properties of the objects under investigation. From what they notice during the investigation phase, students begin to make *conjectures*—unproven statements—based on their preliminary observations of geometric objects. After students make a conjecture, it is important that they either *justify* (explain) the correctness of their conjecture using evidence or revise the conjecture based on new information. The final step in the geometric reasoning cycle is for a student to *prove* their conjecture via a logical argument that demonstrates that the conjecture is always true.

This chapter explores elements of the geometric reasoning cycle and how these processes equip students to engage in modeling real-world and math phenomena in geometry and trigonometry. Specifically, this chapter highlights the themes of exploring geometric relationships (investigation and conjecture), generalizing (justification and proof), and geometric modeling. Throughout the chapter, a focus on the use of technology in teaching geometry and trigonometry to support student thinking is discussed. Examples are provided to illustrate practical tasks that can be used in the teaching of geometry and trigonometry. Finally, two sample tasks are presented to highlight and synthesize the main themes from this chapter.

2. Exploring Geometric Relationships

2.1. Investigating and Conjecturing. Geometric reasoning begins with investigation and conjecture. A big idea offered by Sinclair, Pimm, and Skelin (2012b, 22) is that "geometry is about working with variance and invariance, despite appearing to be about theorems." Through exploring invariance, students try to determine what does not change about an object when other things around it change. During these investigations, students come to understand why theorems are true for all cases satisfying the theorems' given conditions. Driscoll et al. (2007) add "investigating invariants" as one of four geometric habits of mind that foster geometric thinking. Through investigations, students get to explore geometric relationships that lead them to offer conjectures—a key element to developing students' ability to reason and sense-make (McCrone et al. 2010).

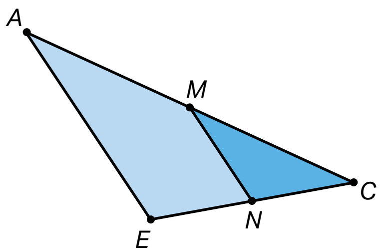
For example, imagine students are asked first to draw two lines that intersect and then to make a conjecture about the resulting angles. If students draw a picture, they may notice that the opposite angles are always congruent regardless of the size of the angle created at the point of intersection of the two lines. The invariant is the vertical angle measures being congruent even as the particular lines and the ways that the lines intersect vary in the plane. As another example, consider students being asked to investigate rotations of a figure. They may notice that rotations are transformations in which distance and angles remain invariant. However, if they investigate dilations, they may observe that angle measure remains invariant but that distance does not. Further exploration would highlight that while distance varies for dilations, the ratios of corresponding side lengths remain invariant.

Exploration of invariance can be aided by dynamic geometry environments (DGEs), technological tools that allow students to investigate geometric relationships through dragging points, lines, circles, and other elements constructed in the DGE. As Hollebrands and Smith (2009, 222) note, "When an element of such a construction is dragged, the geometrical object is modified while all the geometric relationships used in its construction are preserved." While students in the vertical angles example above might investigate and conjecture based on a single picture they drew, DGEs afford investigating students an infinite number of cases through dragging. The power of the tool lies in the fact that the DGE highlights the invariant properties as the element selected for dragging varies.

Consider the investigation illustrated in figure 2: students draw a triangle, construct a line through the midpoint of one side of the triangle that passes through the midpoint of another side of the triangle, and then come up with as many conjectures as they can.

Slope $\overline{NM} = -1.52$ $\quad MN = 5.89$ cm $\quad$ Area $\Delta MCN = 14.67$ cm^2

Slope $\overline{AE} = -1.52$ $\quad AE = 11.77$ cm $\quad$ Area $\Delta ACE = 58.69$ cm^2



$$\frac{MN}{AE} = 0.50$$

$$\frac{\text{Area } \Delta MCN}{\text{Area } \Delta ACE} = 0.25$$

Figure 2. Investigating the midsegment of a triangle

The segment connecting the midpoints is called a *midsegment*, but not knowing that term does not hinder students from engaging in this investigation. Students may observe that several things remain invariant when they drag points *A*, *C*, or *E*: the midsegment is parallel to the third side of the triangle, the midsegment is half the length of the third side of the triangle, the area of $\triangle ACE$ is proportional to the area of $\triangle MCN$, the ratio of all the sides is $1:2$, and the ratio of the areas remains constant $(1:4)$. Prerequisite to entering this investigation is knowledge of the definition of *midpoint*, but students do not need additional precise vocabulary to explore the geometric relationships in the figure and to develop their own conjectures. In fact, the investigation and conjecture processes can themselves develop students' vocabulary as new terms are opportunistically introduced during lessons.

Investigations can also be used to develop trigonometric ideas. Weber (2008) suggests having students construct a unit circle on a coordinate plane, drawing a ray from the origin extending out and intersecting the unit circle, and having students determine the point of intersection of the ray and the circle. Through repeating this several times in the investigation, students begin to realize that the cosine of the angle is the *x*-coordinate of the intersection point and that the sine of the angle is the *y*-coordinate. Weber (2008, 147) argues taking a geometric approach to developing the trigonometric functions "can lead students to understand trigonometric operations as functions," which is very beneficial for linking algebra and geometry and creating a strong foundation for trigonometry. He also notes that this function-based understanding can help students evaluate a function for a given angle.

Almost any theorem in geometry can be turned into an investigation. Investigating invariants can be as simple as asking students to explore any of the following: What is the interior angle sum in a triangle? What do you notice about the angle measures formed by two parallel lines intersected by a transversal? What are the properties of a parallelogram? What is the measure of an inscribed angle of a circle? What do you notice about two chords equidistant from the center of a circle? What is the relationship between the volumes of a cylinder and a cone with the same base and height? By exploring these and other geometric relationships, students become active in constructing their knowledge of geometry. Dragging objects in a DGE provides opportunities for students to more readily observe invariances, which can lead to students creating well-reasoned conjectures (Battista 2008; Leung 2008). The tools used in investigations help students believe the conjectures they make, but, as discussed later, it is still important for students to justify why their conjectures are correct.

2.2. Defining. Another important aspect of exploring geometric relationships is *defining*. As Sinclair, Pimm, and Skelin (2012b, 36) note, "Working with and on definitions is central to geometry." Defining allows students to notice geometric structure and classification, which highlight relationships among geometric objects. Definitions highlight properties that must hold for an object to be a member of the set being defined, while the very act of defining requires students to determine the key features of the object being defined.

Students need experiences in which they create definitions (de Villiers, Govender, and Patterson 2009) and classify figures based on their definitions. Students might start with a definition of *rectangle* as "a quadrilateral that has four right angles." They could then explore whether other definitions would give an equivalent set and therefore also be sufficient as a definition for *rectangle*. A good definition can be written as a biconditional, meaning the definition must apply to all the objects in the set and that the definition results in all objects for which the definition applies being represented in the set. For example, consider the definition of *rectangle* as "any quadrilateral with congruent diagonals." It is true that rectangles have congruent diagonals, but isosceles trapezoids fit this definition too and are outside of the set we are defining. If the definition for *rectangle* instead was "having congruent diagonals that are perpendicular bisectors of each other," isosceles trapezoids would not be included. However, this definition only applies to squares and not to rectangles that are not squares. Defining *rectangle* as "a parallelogram with one interior right angle," by contrast, would apply to every rectangle, and any figure constructed based on this definition would have to be a rectangle. Definitions tell students which shapes and objects are and are not included in the set being defined, which is important because it allows students to better understand inclusion and exclusion criteria for a set. This way of thinking about definitions requires that students notice the structure of the geometric objects being defined and classified.

CollegeBoard

Digital SAT Suite Connections

Geometry is all about modeling the world around us, and knowledge of geometry helps lay the foundation for further achievement in math. Skills, knowledge, and concepts learned in the study of geometry are called on in questions in the Geometry and Trigonometry content domain (for the PSAT 8/9, the Geometry domain) but are also woven into questions in the Algebra and Advanced Math domains, where geometric objects are sometimes used as contexts for building functions or modeling real-world scenarios. Geometry content on the digital SAT Suite is covered in secondary-level courses from grade 6 through high school and is tested at appropriately challenging levels. Trigonometry content is tested on the SAT and the PSAT/NMSQT and PSAT 10, and questions about trigonometry in the digital SAT Suite assess skills and knowledge typically taught and learned only in high school courses.

Test questions in the Geometry and Trigonometry content domain involve applying skills and knowledge in finding areas, perimeters, volumes, and surface areas; using concepts and theorems related to lines, angles, and triangles (PSAT 8/9 includes triangle angle sum theorem only); solving problems using right triangles (SAT, PSAT/NMSQT and PSAT 10 only); solving problems using right triangle trigonometry (SAT, PSAT/NMSQT, and PSAT 10 only); calculating using sine and cosine (SAT only); and using definitions, properties, and theorems relating to circles (SAT only). These test questions vary in difficulty from easy to very hard and allow students to demonstrate problem-solving skills and knowledge using a variety of solving strategies.

It can also help students to examine nonexamples when exploring definitions. Comparing examples and nonexamples can highlight attributes and properties of the objects being defined, leading to a more sophisticated understanding of the objects. The examples and nonexamples of reflections in figure 3 informally show that reflections are "flips" over a line, but further analysis highlights that the image of a reflection is a mirror image of the preimage (i.e., a congruence transformation or isometry). Students can observe that reflections are mappings that maintain the same size and shape, that the orientation of the preimage changes, and that each point in the image and preimage is equidistant from the same line (i.e., the line of reflection is the perpendicular bisector of each point in the preimage and the corresponding point in the image).

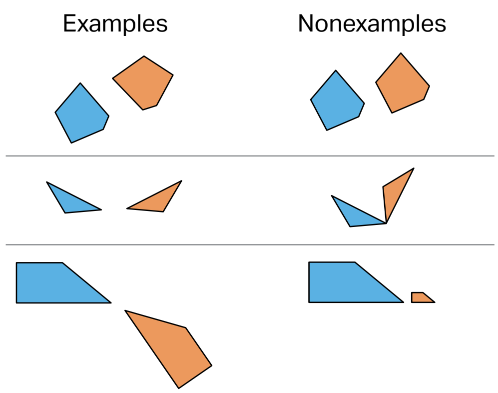


Figure 3. Examples and nonexamples of reflections

To summarize, exploring geometric relationships is an important process in learning geometry. Students construct their knowledge of geometry through discovering important geometric relationships. Investigation and conjecturing provide informal opportunities for students to make sense of math prior to generalizing it for all cases. DGEs support student exploration by allowing them to drag certain features of a figure or shape and observe how the other elements respond dynamically to the dragging. Through their observations, students come up with reasoned conjectures. DGEs can also support students in developing precise definitions of geometry objects that can then be used in communicating ideas and justifying and proving conjectures. These key skills, developed in the geometry classroom, provide a strong foundation for success in college-level math.

2.3. Generalizing. *Investigating* involves making observations and then exploring those observations to make reasoned *conjectures*. *Generalizing* involves abstracting beyond a specific case, or even beyond what can be observed using a DGE, and making an argument for why something is true for all cases. Driscoll and colleagues (2007, 12, and 2009, 162; emphasis in original) describe *generalizing* as "wanting to understand and describe the 'always' and the 'every' related to geometric phenomena." To promote generalizing, they suggest considering such questions as "Does this happen in every case?", "Why would this happen in every case?", "Have I found *all the ones* that fit this description?", "Can I think of examples when this is not true, and, if so, should I then revise my generalization?", and "Would this apply in other dimensions?" As an example, students may be able to reflect the point $(4, 7)$ over the x-axis to arrive at $(4, -7)$. However, to determine what happens to any point that is reflected over the x-axis, they would have to generalize to all cases such that the point (x, y) reflected over the x-axis would be mapped onto $(x, -y)$.

Consider a problem offered by Driscoll and colleagues (2007, 47), represented in figure 4.

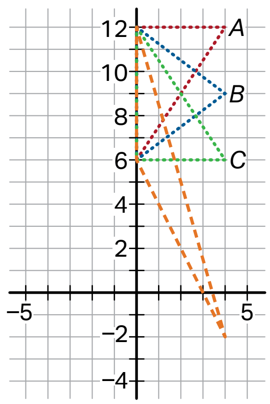


Figure 4. Area of 12 problem

Two vertices of a triangle are located at $(0, 6)$ and $(0, 12)$. The triangle has an area of 12. What are all possible positions for the third vertex? How do you know you have them all? Students are often able to find one answer, typically $(4, 6)$ or $(4, 12)$. When they consider, perhaps with prompting, whether they have all the possible positions, they may begin to add additional points, perhaps $(4, 9)$ or $(-4, 6)$ or $(-4, 12)$. Then they may become convinced that even a point such as $(4, -2)$ would work, which it does. Eventually, through considering the question "Have I found them all?", students understand that any point on the line $x = 4$ or $x = -4$ would provide a solution to the problem. To ensure students understand the geometric relationships in the problem, it is important to have them justify why any point on the line $x = 4$ or $x = -4$ would result in an area of 12. Since the base is 6 units, any triangle with a height of 4 will result in $\text{Area}_{\triangle} = \frac{1}{2} \times b \times h = \frac{1}{2} \times 6 \times 4 = 12$. The key concept in this task is that *all* triangles have the same area when they have the same base and height. This understanding is important not only in this specific case but also for generalizing to other problems involving the area of figures.

2.4. Justification and Proof. After students have experience investigating and conjecturing, it is important that they create a geometric argument for why a given conjecture is true in all cases. Once a generalization is proposed, it is necessary for them to consider other geometric properties in order to offer a compelling case for why the generalization holds. Thus, *justification* requires drawing on previously established geometric knowledge, including axioms, theorems, and definitions, to establish a clear argument in support of the generalization.

Take the previous example in which students were asked to draw two lines that intersect and then make a conjecture about the resulting angles. Once students conjecture that vertical angles are congruent, they can be encouraged to justify why this result holds. Using what they know about supplementary angles, they can come up with a convincing argument that the vertical angles have equal measure and are therefore congruent. Alternatively, using a transformations-based approach, they may use what they know about rotations to show that if they rotate one of the angles 180 degrees about the point of intersection, each line rotates onto itself and lands on the other angle. Therefore, since rotations preserve angle measure, the two angles have equal measure. These justifications can initially be informal but will lead to deductive reasoning and eventually to formal proof.

Geometric reasoning can be cultivated through the act of justifying why generalizations are true. As de Villiers (2010) notes, deductive proof is not only about verifying an important result but also about explaining why a theorem is true. McCrone et al. (2010, 1) add that "developing and evaluating deductive arguments (both formal and informal) about figures and their properties that help make sense of geometric situations" are key elements of building students' reasoning skills. Both formal and informal *proof* is about establishing generality, which organizes what is known and understood into a logical and compelling argument.

It is important for students to understand that once a statement is proved to be true, it is true for all possible cases of the statement. For example, a proof of the Pythagorean theorem, $a^2 + b^2 = c^2$, in a right triangle with legs of length a and b and hypotenuse of length c would mean that any time there is a right triangle, the relationship $a^2 + b^2 = c^2$ is true. Conversely, students need to know that to prove something is not true, they just need to find one, and only one, counterexample that shows that the hypothesis is true but the conclusion is false. For example, consider the statement If $x^2 = 9$, then $x = 3$. This is false, as the counterexample of -3 illustrates: If $x = -3$, then $x^2 = 9$, but the conclusion, $x = 3$, is false. Sinclair, Pimm, and Skelin (2012b, 51) proposed an activity to have students explore such counterexamples. They had students determine which of the following conjectures were true and which needed modifying; if any needed modifying, students were asked to provide a counterexample.

A. The perpendicular bisectors of the sides of a triangle always meet inside the triangle.
B. A triangle always has a longest side.
C. Through any three points you can always draw a circle.
D. Through any four points you can always draw a circle.
E. The diagonals of a quadrilateral always intersect.

Not only is this exercise good practice in generating counterexamples, but it also helps students consider the necessary and sufficient conditions needed to revise conjectures.

Constructing viable arguments and justifying why conjectures and theorems are true are hallmarks of geometric thinking. Students will often accept a result for one case but struggle to both generalize it and explain why it is true for all cases. Asking students to justify why a result is true serves to foreground their thinking and can guide the questions teachers ask if students need to revise their justifications. Ellis, Bieda, and Knuth (2012, 9) describe the numerous roles of proof: verifying whether a statement is true, providing insight as to why a statement is true, developing a new theory, communicating math knowledge, and supporting the use of precise math language. Throughout high school, students should gain experience with each of these roles in order to develop deep, connected knowledge of geometry.

2.5. Geometric Modeling. The geometric reasoning cycle's four steps—investigation, conjecture, justification, and proof—are central processes for developing students' understanding of geometric concepts, but being able to apply that reasoning is also a critical competency in the learning of geometry. As such, geometric modeling is important for applying knowledge of geometry to real-world situations. Specifically, *modeling* revolves around tackling a real-world scenario, building a mathematical model of the scenario, using math on the model to come to a conclusion, and interpreting results in the context of the scenario (McCrone et al. 2010). Geometric reasoning is developed so that it can be applied to modeling real-world and mathematical phenomena. As Freudenthal (1973, 407) put it, geometry is "one of the best opportunities which exist to learn how to mathematize reality."

Modeling is central to the high school math curriculum (e.g., CCSSM, NMAP) and one of the Standards for Mathematical Practice highlighted extensively throughout the CCSSM document (NGA Center for Best Practices and Council of Chief State School Officers 2010). Modeling is an iterative process connecting a real-world problem to math and math back to the real-world problem. This is standard practice in geometry: taking the real-world problem, turning it into a geometric model through

College and Career Readiness

The *College Board National Curriculum Survey Report 2019* offers strong indications that geometry and trigonometry skills and knowledge are important college and career readiness prerequisites.

The sample of 268 postsecondary math faculty rated the following geometry and trigonometry skill/knowledge elements assessed on the digital SAT Suite on a four-point scale, with 4 meaning "very important":

- Solve problems using area and volume formulas (3.35), special right triangles (2.99), the Pythagorean theorem (3.39), theorems of triangle similarity and congruence (2.69), circle theorems (2.52), logical reasoning and mathematical proofs (2.62), and trigonometry relationships, including sine, cosine, and tangent (2.50)

Ratings for all these skill/knowledge elements met or exceeded the threshold of 2.50 established as indicating a given skill/knowledge element was considered important as a prerequisite for readiness for postsecondary math.

For more information on College Board's national curriculum survey and its results, see the general introduction to this collection.

imagery and diagrams, working to find a solution, and interpreting the solution in terms of the real-world problem. For example, consider building a theater stage for a school play. Stages can be elevated to give the audience a better view, but too much of a rise can pose a safety hazard for the performers. If provided with measurements for the stage and an acceptable range for the angle of elevation, a student could model the situation to determine a proper angle to build the stage to ensure the entire audience can see the show without risking the performers' safety.

As McCrone and colleagues (2010, 77) observe, "Knowledge that mathematics plays a role in everyday experiences is very important. The ability to use and reason flexibly about mathematics to solve a problem is equally valuable. These two come together in mathematical modeling to solve real-world problems." Knowing that math can be used in real life and having flexible mathematical solution strategies are both important for modeling. McCrone et al. further argue that modeling clears pathways for new math learning and opens space for students to consider multiple solution methods as well as to think about the strengths and limitations of particular solution strategies. Even more importantly, students begin seeing math in their everyday lives and get to use the math tools they have learned to solve problems that are meaningful to them in the real world.

Modeling in geometry also supports growth in students' visual thinking.[1] Many examples of modeling require students to draw new diagrams to represent a situation, imagine geometric objects in space, or consider dynamic relationships in a static figure. For example, given circle O with secant line ℓ intersecting the circle at points Z and Y in figure 5, consider what happens to ZY if you move ℓ around and leave Z fixed.

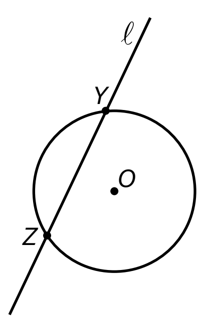


Figure 5. Example of dynamic relationship in static figure

In high school, visualization skills extend from 2D and 3D representations to spherical surfaces, perspective drawings, and cross sections of solid objects (which result in conic sections). Students need experience

[1] Students who are blind or visually impaired can still access geometry content through physical models they can touch and explore. Additionally, some software has been developed to help students with visual impairments access dynamic computer-based environments (Figueira et al. 2015).

representing 3D figures with nets and then using the nets to find surface area in real-world problems. DGEs and other web-based technologies (e.g., applets, Desmos, Tinkercad, SketchUp, virtual manipulatives) help students visualize and manipulate the geometric objects they are representing.

Modeling tasks can range from daily lessons to an activity returned to throughout a unit to a project students complete. For example, McCrone et al. (2010) present a Clearing the Bridge problem that invites students to explore why some tractor trailer drivers get stuck under bridges despite maximum clearance signs posted on roadways. McCrone and colleagues present a series of five tasks to help students reason through the situation: building a model, identifying variables, investigating "dangerous height," making connections to algebra and trigonometry, and making connections to symbolic representations. In this extended modeling task, students create representations of the situation, use their knowledge of triangles and trigonometry, develop new geometry learning, and interpret their solution in terms of an authentic scenario.

A shorter example of a modeling task for middle grades involves having students consider the logistics of effectively mailing packages using shipping cartons assembled from flat pieces of cardboard, as shown in figure 6.

> The Helping Heroes Organization is dedicated to shipping care packages to soldiers. Your task will be to determine what size care packages should be sent to cost-effectively ship the care packages containing the most items. The packages are made from $8\frac{1}{2}$-by-11-inch pieces of cardboard that are folded into rectangular prisms.

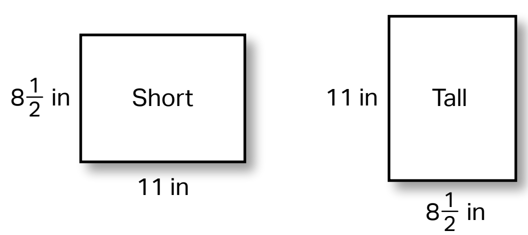


Figure 6. Mailing packages

This task has students exploring the relationship between surface area and volume—specifically, that two figures could have the same surface area and different volumes or, conversely, two figures could have the same volume and different surface areas. Students engage in a discovery-based task to determine these relationships, which are counterintuitive to how most students initially think about surface area and volume. Visually, they have to consider how the side lengths of their original cardboard change as it is folded into a rectangular prism. (Note that in this case, one would still need to add a top and bottom to the package.) In this example, students investigate a situation in which the surface area is the same for both the short and tall packages pictured above but the volume of the short rectangular prism is larger. Students first have to determine which package results in more space in which to ship items and then consider why these conditions result in a larger volume.

Scale drawings are important for modeling, as they allow students to solve problems by creating ratios for corresponding lengths between an original and a scaled object. An understanding of ratio and proportion is essential prerequisite knowledge for exploring scale factors and solving for missing side lengths in scaled drawings. Scaling is a vital concept for understanding similarity and transformations and in careers such as architecture, design, and engineering.

2.6. Measurement. A key component of being able to model in math is measurement. Modeling in geometry involves being able to apply geometric properties and measurement. When students apply the theorems and definitions they know to solve problems that require finding angle measures, side lengths, perimeter, area, surface area, or volume, measurement is a critical element.

Recall from our earlier discussion the NAEP test question that only 5 percent of students correctly answered (figure 7). This problem involves not only linear measurement but also measurement in three dimensions. To answer the question, students have to use their knowledge of reflections, perpendiculars, area, and right triangles. Making sense of this question relies on visually thinking about segments that cannot be drawn on paper and requires spatial reasoning to draw triangles DBE and EFG and to label what is known and what needs to be known to find the area of triangle EFG, which is $12\sqrt{2}\,\text{cm}^2$.

Fold the card provided with this test book along line AC so that point D coincides with point E. Open the card so that segment BE is perpendicular to segment BD, as shown below, and keep the card in this position to answer the question.

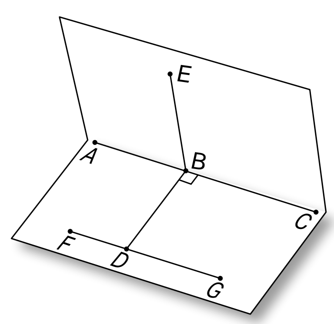
The lengths of some of the line segments on the card are given below.

$AC =$ 12 centimeters	$BD =$ 3 centimeters
$BE =$ 3 centimeters	$FG =$ 8 centimeters

What is the area of triangle EFG ?

Answer: _________ cm^2

Show how you found your answer. (Work done on the card will not be scored.)



Note: Figure not drawn to scale.

Figure 7. NAEP released item (adapted from NAGB 2013)

Other measurement problems allow for students to navigate flexibly through several viable solution pathways. For example, the figure in Lawson and Chinnappan's (2000) Problem-Solving Task (figure 8), which involves properties of tangent lines to a circle, could be approached as consisting of three 30°–60°–90° triangles, the relationships among the sides of which could be used to find a solution. Alternatively, students could draw on their knowledge of similar triangles ($\triangle ACE \sim \triangle DCE \sim \triangle ADC$ by angle-angle similarity) to create ratios for the sides, initially using right triangle trigonometry and then solving for the missing segment. Again, students are drawing on the theorems and formulas they have previously explored, but this type of problem allows students to find their own unique solution strategies.

AE is a tangent to the circle, center C.

AC is perpendicular to CE, and angle DCE has a measure of 30°.

The radius of the circle is equal to 5 cm.

Find AB.

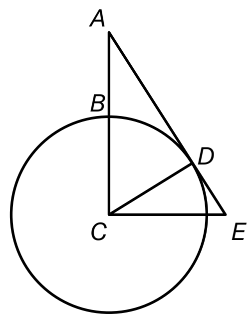


Figure 8. Problem-Solving Task (adapted from Lawson and Chinnappan 2000)

In summary, skill in and knowledge of measurement can support students' geometric modeling. Modeling, in turn, both allows students to showcase their existing knowledge of geometry and helps them create new knowledge. Through modeling, students are able to engage in investigation, conjecture, justification, and proof. Modeling frequently involves finding the measures of geometric objects, an activity that relies heavily on students' understanding of properties and theorems. Modeling tasks also often help students make connections to new and existing ideas and to other math domains, especially algebra, trigonometry, and probability.

3. Putting It All Together

This section presents two geometry tasks that bring the ideas in the chapter together. The first task highlights how students can combine the geometric reasoning cycle with modeling to develop new geometric concepts. The second task illustrates how students can use their prior knowledge to generalize a new result modeling a real-world scenario.

3.1. Pizza Parlor Proximity Problem. The geometric reasoning cycle can be used to develop new understandings to model real-world scenarios. Consider the Pizza Parlor Proximity Problem (National Council of Teachers of Mathematics 2008, figure 9), in which a pizzeria owner in the fictional town of Squaresville wants to ensure that when a customer orders pizza, the nearest of their pizzerias takes the call, thereby minimizing delivery time.

> You are the owner of two pizzerias in the town of Squaresville. To ensure minimal delivery times, you devise a system in which customers call a central phone number and are then transferred to the pizzeria that is closest to them. This map of Squaresville shows the position of the two pizzerias. You need to divide the town into two regions so that customers order their pizza from the closest pizzeria.

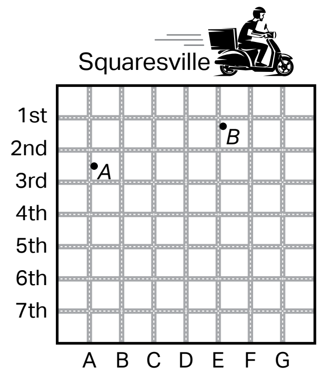


Figure 9. Pizza Parlor Proximity Problem (adapted from National Council of Teachers of Mathematics 2008)

Students start by investigating which calls are transferred from a central number to pizzerias A and B, using the grid in the figure as a mathematical model. Initially, students may go block by block to determine which pizzeria accepts each call. For example, if a person called from the corner of A and 1st Streets or B and 1st Streets, their pizza would come from pizzeria A, but if a person called from G and 1st Streets, their pizza would come from pizzeria B. Students could fill in the entire grid with "A" and "B" simply by reasoning about which pizzeria is closer. Once they complete that task, they will realize that there is an imaginary line dividing Squaresville: every call to the left of that line would go to pizzeria A and every call to the right of it would go to pizzeria B. Students may then create a more abstract mathematical model of the town by eliminating the grid and creating a segment connecting points A and B. At this juncture, they may realize that the line that partitions the town is equidistant from points A and B. Their investigation leads them to conclude that this line passes through the midpoint of $\overline{AB}$ and that it is perpendicular to $\overline{AB}$.

Based on their investigation, students may use a DGE to further explore what they are noticing. After constructing a segment, the midpoint of the segment, and a perpendicular line through the midpoint of the original segment, students may begin exploring what remains invariant as points are dragged along the perpendicular bisector. Alternatively, they may construct a segment on paper and fold the paper so the endpoints meet, creating a midpoint and a perpendicular line, and then use a ruler to further investigate the relationships they are noticing. Eventually, students may conjecture that any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment. In other words, in terms of the real-world situation, if anyone calls in from a point on the perpendicular bisector of $\overline{AB}$, the call could be routed to either pizzeria and achieve minimal delivery time. However, any call originating to the left of the line should be transferred to pizzeria A and any call to the right of the line to pizzeria B. This problem highlights the iterative process of going from the real-world scenario back to the math model and then back to the real-world scenario.

Students discover this theorem, which we might refer to as the perpendicular bisector theorem, from their own intuitive exploration of the Pizza Parlor Proximity Problem. Teachers should then guide the students to justify why their conjecture is true. This justification should be based on students' prior knowledge of definitions (e.g., *midpoint*, *perpendicular*) and of either triangle congruence or transformations and reflections (which preserve distance). Then students can prove this result is true for any point along any perpendicular bisector. Through this one task, students use geometric reasoning to investigate, conjecture, justify, and prove a new result based in a real-world situation.

3.2. Modeling and Discovering the Law of Sines. Some modeling examples have students discover new mathematical results and generalize them. Students who only have knowledge of right triangle trigonometric functions can discover the Law of Sines through the Electric Turbine Task (adapted from Paniati 2009, figure 10).

> An electric company has found a location for a new offshore wind turbine in the middle of the ocean. The turbine is going to power two beach towns 6 miles apart. The electric company needs to lay underwater cables to send the collected energy to both towns. Based on the picture below, how many miles of cable does it need to connect the wind turbine to Town A and Town B?

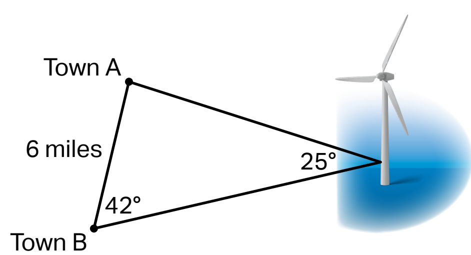


Figure 10. Electric Turbine Task (adapted from Paniati 2009)

In this task, students have to find the length of underwater cable needed to connect two towns to a wind turbine and are given the distance between the two towns and two angles of the triangle connecting Town A, Town B, and the wind turbine. Initially, students reason about measurement—specifically, how to find the missing side lengths when the triangle is not right—which will eventually lead them to construct a perpendicular line from Town A to the opposite side of the triangle. Then, using what they know about right triangle trigonometry, they can find the length of the newly constructed height. Using the measure of that height, they can, through setting up several more equations, find the lengths of the other two sides of the triangle, representing how much cable is needed.

Once students have solved the Electric Turbine Task, they can be asked to find a generalization (i.e., formula) that relates the angle at Town B, the angle at the turbine, side *b*, and side *t* (figure 11).

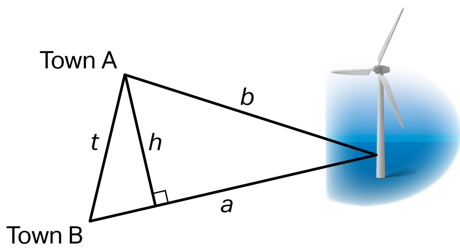


Figure 11. Follow-up to Law of Sines Discovery

Using the same reasoning as above, they can show that $\sin B = \frac{h}{t}$ and $\sin T = \frac{h}{b}$, which implies $h = t \times (\sin B) = t \times (\sin T)$. In doing so, they can derive that $\frac{\sin B}{b} = \frac{\sin T}{t}$. In this task, students use their knowledge of geometry and measurement to construct a new formula, and in deriving the new formula they make connections among geometric ideas to generalize and model in geometric contexts.

Conclusion

Geometry is often thought of as a course in which deductive reasoning is built, algebra can be applied to solve geometric modeling problems, and important formulas (area formulas and the Pythagorean theorem) are learned. However, geometry should not merely be an isolated one-year course in high school; rather, geometric thinking should be infused into algebra courses to provide strong connections to the concept of invariance and to provide properties of figures that can be used to solve algebraic equations. Geometry should also be taught as a way to develop trigonometric ideas and spatial thinking, which are topics important for a strong foundation for first-year college math courses and later for advanced college math. While it is important to do all those things in geometry, as this chapter has laid out, it is also important for students to have opportunities to do those things while investigating, conjecturing, justifying, and proving. Each of the four steps of the geometric reasoning cycle helps build students' geometric reasoning, reasoning they can then apply to other branches of math. Facility with the reasoning cycle is enhanced when students have opportunities to informally explore geometry and thereby develop key geometric concepts. DGEs offer robust opportunities for students to explore by providing limitless variations of geometric objects that allow them to visualize relationships more effectively than is possible using static images. Through this exploration, students are better equipped to generalize geometric relationships, which assists in justification and proof. Finally, after exploration and generalization, students have the tools they need to model mathematical and geometric phenomena, which prepares them to model the world around them. ❖

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